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A calculation of the plane wave string hamiltonian from $\mathcal{N}=4$ super-Yang-Mills theory

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Abstract: Berenstein, Maldacena, and Nastase have proposed, as a limit of the strong form of the AdS/CFT correspondence, that string theory in a particular plane wave background is dual to a certain subset of operators in the $\mathcal{N}=4$ super-Yang-Mills theory. Even though this is *a priori* a strong/weak coupling duality, the matrix elements of the string theory hamiltonian, when expressed in gauge theory variables, are analytic in the 't Hooft coupling constant. This allows one to conjecture that, like the masses of excited string states, these can be recovered using perturbation theory in Yang-Mills theory.
In this paper we identify the difference between the generator of scale transformations and a particular U(1) R-symmetry generator as the operator dual to the string theory hamiltonian for nonvanishing string coupling. We compute its matrix elements and find that they agree with the string theory prediction provided that the state-operator map is modified for nonvanishing string coupling. We construct this map explicitly and calculate the anomalous dimensions of the new operators. We identify the component arising from the modification of the state-operator map with the contribution of the string theory contact terms to the masses of string states.

Keywords: 1/N Expansion, Penrose limit and pp-wave background.

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Contents

1. Introduction

"How do strings behave for strong (world sheet) coupling?" This question is an interesting one and its answer could lead to enormous progress in string theory. The various dualities of string theory were important sources of insight. Perhaps the closest we have come to answering this question in a particular context comes from the AdS/CFT duality which seems to suggest that string theory on an AdS space at large world sheet coupling has a description in form of perturbative $\mathcal{N} = 4$ Yang-Mills theory. Proving this statement turns out to be unexpectedly hard due to the nonlinearities of the world sheet action.

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It is clear that the above setup is just a limit of the strong form of the AdS/CFT duality. Fortunately, other limits can be found in which the action becomes quadratic in light-cone gauge. An interesting proposal was put forward in [1], relating a particular sector of an SU(N) gauge theory to string theory in a plane wave background. The plane wave considered in [1]

$$ds^2 = -dx^+dx^- - \mu^2(x^2+y^2)dx^{+2} + dx^2 + dy^2 \qquad F_{+1234} = F_{+5678} = \mu \tag{1.1}$$

was obtained as a Penrose limit of $AdS_5 \times S^5$. On the gauge theory side the limit focuses on the set of operators with R-charge J which, for fixed Yang-Mills coupling, scales with N such that $J/\sqrt{N}$, as well as the difference between the dimension of operators and their R charge are fixed. This setup came to be known as the BMN limit. The details of this limit imply that, while this proposal is a limit of the strong form of the AdS/CFT duality, it cannot be reached by a sequence of theories each at infinite N. It is therefore clear that this duality probes finite N regions of the initial AdS/CFT duality.

Even though it is sourced by a nontrivial RR field, the background (1.1) is substantially simpler that the original $AdS_5 \times S^5$. It turns out that, for a string in such a background, the world sheet theory is free and thus there is no difficulty in quantizing it and finding the free spectrum [4, 5]. The details of the BMN limit relate various operators on the string theory side with operators on the gauge theory side. Of particular relevance for this paper is the hamiltonian P^- which, for the Euclidean gauge theory, turns out to be given by the difference between the generator of scale transformations and the R-charge generator J [1]:

$$P^- = \Delta - J\,. \tag{1.2}$$

From the considerations in [1], it follows that this relation holds at tree level.

At first sight this limit does not seem to bring us any closer to understanding how a string theory can be related to a weakly coupled field theory, since the Yang-Mills coupling $g_{\rm YM}$ is kept fixed while the number of colors is infinite and thus the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ is infinite. However, a closer inspection of string theory results reveals that they have a weak Yang-Mills coupling expansion as well. This expansion also implies that the effective coupling is not the 't Hooft coupling, but $\lambda' = \lambda/J^2$. In string theory variables this is $\lambda' = 1/(\mu p^+ \alpha')^2$. In the BMN limit this quantity is arbitrary and can be taken to be small. It appears therefore that in the BMN limit the gauge theory develops an effective coupling in tems of which it is weakly coupled. In [2] we checked that a finite N and J version of λ' is indeed the effective field theory coupling to two loops and conjectured that this is the case to all loop orders. Based on this conjecture we proved that the anomalous dimensions of operators conjectured to be dual to string states have the precise value predicted by string theory. Our conjecture was later proven in [3] where a different computation of the anomalous dimensions was presented. This beautiful proof, based on superconformal invariance and using equations of motion, represents a strong test of the validity of equations of motion at the quantum level and from certain points of view suggests that the matching of non-BPS operators in the particular subsector of the $\mathcal{N} = 4$ SYM might be a consequence of the superconformal symmetry.

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Recovering the exact tree level string spectrum seems like a small miracle, since there does not exist any *a priori* reason for the gauge theory perturbation theory to give the full answer. It is not clear why the series representation of the anomalous dimension of the operators considered should converge at all. Nevertheless, the results of perturbative computations in gauge theory seem to suggest that string theory in the plane wave background has a good chance of having a description in the form of a field theory which is effectively weakly coupled.

Given the success in matching gauge theory operators with string states, it is natural to wonder whether string interactions have a perturbative gauge theory representation. The simplicity of (1.1) allows one to explicitly study interactions, splitting and joining of strings. In [7] the 3-string hamiltonian was constructed, together with the order g_s corrections to all dynamical symmetry generators.

However, the pure λ' expansion in gauge theory does not carry any information about string interactions. This is clear, since in string theory variables λ' is independent on the string coupling constant. On the string theory side there exists a dimensionless, arbitrary parameter, which is related to the string coupling:

$$g_2 = g_s(\mu p^+ \alpha')^2 \tag{1.3}$$

which in gauge theory variables is

$$g_2 = \frac{J^2}{N}\,. \tag{1.4}$$

The appearence of a $1/N$ implies that this parameter describes the genus expansion in gauge theory. This is indeed the case, as it has been shown in [6, 9].

Based on gauge theory computations, the authors of [6] were led to conjecture a certain expression for the matrix element of the string hamiltonian between a 1- and a 2-string state. The essential test for this conjecture was the recovery of the gauge theory prediction for the masses of string states at order g_s from a "unitarity computation":

$$\Delta E_n = \sum_{n\neq m} \frac{|V_{nm}|^2}{E_n - E_m}\,. \tag{1.5}$$

On the string theory side this corresponds to a 1-loop computation.

Subsequent string field theory analysis performed in [8] revealed that this conjecture does not hold, and the correct matrix elements of the string hamiltonian were computed.[1]

The issue of recovering the interactions of the string in a plane wave background from perturbative Yang-Mills theory was discussed by various authors: n-point functions were discussed in [15, 16, 18] where some tests of the conjectured matrix elements of string hamiltonian were proposed. Computations of n-point functions via unitarity sums was discussed in [14]. Some properties of vector operators were analyzed in [10] where an

[1]The original version of this paper was refering to the second version of `hep-th/0206073` where the conjecture was confirmed. This does not imply that the original results were technically incorrect, but only that the operator basis constructed in the original version of this paper is not appropriate for comparison with the correct string theory predictions.

apparent mismatch with unitarity computations was pointed out. A possible solution was suggested in [11]. Deformations of the world sheet action by exactly marginal operators were discussed in [17], where $1/R_{AdS}$ corrections to the masses of the string states were analyzed.

In this paper we will recover the expression for the matrix elements of the string hamiltonian from a perturbative gauge theory computation. In particular, one of the main results will be that equation (1.2) holds to second order in the string coupling constant. In gauge theory these matrix elements arise as two-point functions between appropriately-defined (multi-trace) operators. This result seems to support the suggestion in [19] that string interactions are encoded in two-point functions of appropriately-defined multi-trace operators. An essential ingredient in our derivation are the g_s corrections to the state-operator map of [1]. These corrections are partially fixed by the requirement that in the new basis the scalar product of operators is the identity matrix. However, while some redefinitions may seem more natural than others, the requirement that the scalar product be the identity matrix fixes the state-operator map up to an $\mathrm{SO}(2J+1)$ transformation. This freedom must be fixed by comparing with the string theory predictions.

In the next section 2 we will identify the hamiltonian and present arguments that the correspondence between string states and gauge theory operators must be modified for finite string coupling. We will also discuss the relation between the anomalous dimension matrix, the scalar product of operators and the matrix elements of the proposed hamiltonian. Section 3 is devoted to the computation of the anomalous dimension matrix for operators corresponding to string states created by exactly two creation operators. The results are summarized in equation (3.53). In section 4, by comparing the matrix elements of the gauge theory version of the string hamiltonian with the string theory predictions, we construct the operators that are dual at order g_s to string states, equation (4.19). The anomalous dimension of these operators receives an extra contribution compared to the dimension of the original operators, equation (4.30). We interpret it as arising from the contact terms needed at order g_s^2 for the closure of the symmetry algebra. As a byproduct of this construction we find the free 2-point functions of double-trace operators at genus 1, equation (4.32). As a consistency check we find that the masses of the 2-string states do not receive corrections at order g_s^2. We conclude with a discussion of our results and directions for future work.

Note: while this paper was being written we received the interesting paper [12] discussing mixing of operators. While we agree with the reason for this mixing, we depart from the particular implementation discussed in that paper. We find that recovering the matrix elements of the string hamiltonian from a gauge theory standpoint requires a different choice for the state-operator map, which is related to the one described in [12] by an orthogonal transformation.

Note for the second version: in the original version of this paper we constructed the state-operator map such that the matrix elements of the hamiltonian computed in gauge theory agree with the string theory results at the time the paper was written, hep-th/0206073. The idea of the construction is of course unchanged, but due to subsequent corrections to the string theory computation, this version of the paper presents a different state-operator map. The two sets of operators are related by an $\mathrm{SO}(2J+1)$ transformations mentioned above as well as later in the text. The freedom of performing such transformations was part of the initial construction.

Using the corrected matrix elements of the string theory hamiltonian, the mixing of single- and double-trace operators was later derived also in [22] and [23] . The eigenvalues of the hamiltonian were found in [21].

2. Hamiltonian, operators and mixing

We begin this section with a fast review of the BMN operators and their associated string states. To keep the discussion self-contained and fix our conventions and notations we list here the bosonic part of the $\mathcal{N}=4$ lagrangian with the normalizations we will use in our computations.

$$\mathcal{L} = \frac{1}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\left[D_\mu \Phi^i D^\mu \Phi_i + D_\mu \bar{Z}^i D^\mu Z_i + [Z,\,\Phi_i][\bar{Z},\,\Phi^i] - \frac{1}{4}[Z,\,\bar{Z}]^2 + \frac{1}{2}[\Phi_i,\,\Phi_j][\Phi^i,\,\Phi^j]\right] \tag{2.1}$$

with $i=1,\ldots,4$ and $\mathrm{Tr}[T^aT^b]=\frac{1}{2}\delta^{ab}$. The field Z is complex and corresponds to singling out a $U(1)$ subgroup of the SU(4) R-symmerty group, $Z=\Phi^5+i\Phi^6$, while the fields Φ_i are real.

A generic BMN operator is indexed by a set of n integers $m_i,\ i=1,\ldots,n$ and has the expression:

$$\mathcal{O}^{i_1\ldots i_n}_{m_1\ldots m_n} = \frac{1}{N^{(J+n+1)/2} f(J,n)} \sum_{\substack{l_0,\ldots,l_n=0\\ l_0+\ldots+l_n=J}}^{J} \left(\prod_{k=1}^{n} e^{2\pi i (l_0+\ldots+l_k)\frac{m_k}{J}}\right) \mathrm{Tr}\left[\Phi_0 Z^{l_0} \prod_{j=0}^{n} (\Phi_{i_j} Z^{l_j})\right], \tag{2.2}$$

where $i=1,\ldots,4$ and the function $f(J,n)$ is chosen such that the free 2-point function of these operators is $\frac{1}{|x-y|^{2(J+n+1)}}$. It turns out that this operator has finite anomalous dimension in the BMN limit, to all orders in the Yang-Mills coupling [2], [3]:

$$\gamma_{m_1\ldots m_n} = \sum_{i=1}^{n}\left(\sqrt{1+16\pi^2 m_i^2\lambda'}-1\right). \tag{2.3}$$

These operators (2.2) are conjectured to be in one to one correspondence to the string states in plane wave background, as follows:

$$\mathcal{O}^{i_1\ldots i_n}_{m_1\ldots m_n} \quad\leftrightarrow\quad a^{\dagger i_0}_{-\sum_i m_i} \prod_{j=1}^{n} a^{\dagger i_j}_{m_j} |0\rangle\,, \tag{2.4}$$

where $|0\rangle$ is the lightcone vacuum and $a^{\dagger i}_{m}$ is the creation operator for the i^{th} bosonic string coordinate, with frequency $\mu\sqrt{1+m^2/(\mu p^+\alpha')^2}$. This correspondence will be modified at finite string coupling as we will discuss shortly.

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2.1 The hamiltonian

As described in the introduction, the operator dual to the tree-level string theory hamiltonian P^- is the difference between the generator of scale transformations and the generator of R-symmetry transformations along the geodesics chosen for the Penrose limit. This identification is in fact based on the symmetries of the plane wave which are just a contraction of the 4-dimensional superconformal group. This, together with the fact that the plane wave is an exact solution of string theory, implies that the identification between the string hamiltonian and the difference between the generator of scale transformations and a $U(1)$ R-symmetry generator survives as we turn on the interactions. Thus, we expect the following relation to hold:

$$P^-(g_s) = \Delta - J\,. \tag{2.5}$$

The purpose of this paper it to show that this equation holds to leading order in the string coupling.

To prove that two operators are the same it is enough to show that they have the same eigenvalues. This guarantees that, for any choice of basis for the carrier space of one of them, there exists a choice of basis for the carrier space of the other one such that their matrix elements agree. Thus, from this perspective, proving that two operators are the same involves constructing a map between the spaces they act upon as well as showing that their matrix elements in the basis related by this map, are the same. Thus, to reach our goal it is necessary to construct a map between the string and gauge theory Hilbert spaces, i.e. we have to find the operators dual to multi-string states as well as the appropriate scalar products. We stress that it is not possible to construct this map without performing a detailed comparison of the matrix elements of some operator. As described in the previous subsection, it was argued in [1] that, at vanishing string coupling, the operators dual to 1-string states are the BMN operators (2.2). Explicit gauge theory computations showed that indeed, at vanishing string coupling, the anomalous dimensions of these operators match the tree-level string state masses.

The operators dual to 2-string states can be infered rather easily. In string theory a 2-string state belongs to the tensor product of two 1-string Hilbert spaces. It is therefore natural to identify, at vanishing string coupling, the 2-string states as being the double-trace operators obtained by simply multiplying two BMN operators.

The next step in constructing the map is the identification of a scalar product. On the string theory side the scalar product is fixed to be the scalar product of the 1-string Hilbert space and tensor products thereof for multistring Hilbert spaces. On the gauge theory side, a natural candidate for the result of the scalar product of two operators is the coefficient of their 2-point function. There is, however, a substantial difference between this and the string theory scalar product. While the latter is diagonal to all orders in the string coupling, the former receives various corrections. Thus, the identification between the string and gauge theory scalar products must be reanalyzed order by order g_2 expansion in the gauge theory. This implies that, unless all 2-point functions are trivial, the state-operator map is bound to be modified at finite order in g_s. For example, the free 3-point function of BMN operators of certain types was computed in [9, 6] and found to be nontrivial at order

$g_2 \sim g_s$. It follows that the gauge theory scalar product between operators dual to 1- and 2-string states does not vanish at order $g_2 \sim g_s$. This fact is crucial for the correct identification of the matrix elements of the hamiltonian. We will therefore compute the matrix elements of equation (2.5)

$$\langle \bar{\mathcal{O}}_i [(\Delta - J)\mathcal{O}_{jk}] \rangle = \langle i|P^-|j\rangle|k\rangle \equiv (\langle i| \otimes \langle j| \otimes \langle k|)|V\rangle\,, \tag{2.6}$$

where $\mathcal{O}_i$ and $\mathcal{O}_{jk}$ are the operators[2] dual to the 1- and 2-string states $|i\rangle$ and $|j\rangle|k\rangle$ respectively.

2.2 The anomalous dimension matrix

Using the generator of scale transformations as the hamiltonian produces an intimate relation between the anomalous dimension matrix and the matrix elements of the hamiltonian. Thus, let us proceed with a brief summary of the properties of anomalous dimension matrix.

Consider a set of operators $\mathcal{O}_\Sigma$.[3] An aspect of the renormalization of composite operators is operator mixing: the divergences in 1PI diagrams with one insertion of a composite operator generally contain divergences proportional to other composite operators. Thus, all composite operators must be renormalized at the same time, order by order in perturbation theory. This leads to the following relation:

$$\mathcal{O}_\Sigma^{\text{bare}}(\Phi^{\text{bare}}, g^{\text{bare}}) = \sum_\Xi Z_\Sigma{}^\Xi \mathcal{O}_\Xi^{\text{ren}} \left(Z_3^\Phi \Phi^{\text{ren}}, \frac{Z_1}{Z_3^{\Phi 3/2}} g^{\text{ren}} \right), \tag{2.7}$$

where Φ denotes a generic field, we assumed the existence of a cubic Φ interaction of strength g and Z_3 and Z_1 are the usual wave function and coupling constant Z-factors.

This implies that under scale transformations, the operators $\mathcal{O}_\Sigma$ transform as

$$\mathcal{O}_\Sigma \longrightarrow \Delta \mathcal{O}_\Sigma = \sum_\Xi (d_\Sigma \delta_{\Sigma\Xi} + \gamma_{\Sigma\Xi})\mathcal{O}_\Xi \tag{2.8}$$

were $\boldsymbol{\gamma}$ is the anomalous dimension matrix, and d_Σ is the engineering dimension of $\mathcal{O}_\Sigma$. Its expression, in terms of the renormalization factors $Z_\Sigma{}^\Xi$ introduced above, using dimensional regularization, is given by:

$$\gamma_{\Sigma\Xi}(\lambda) = \lim_{\epsilon \to 0} \left[\epsilon \, \left(Z^{-1}(g_{\text{YM}}, \epsilon) \right)_\Sigma{}^\Psi \frac{d}{d \ln g_{\text{YM}}} Z_\Psi{}^\Xi (g_{\text{YM}}, \epsilon) \right], \tag{2.9}$$

where g_{YM} is the Yang-Mills coupling. We will work mostly at 1-loop level, in which case the expressions simplify considerably. The renormalization constant $Z_\Sigma{}^\Xi$ is

$$Z_{\Sigma\Xi} = \delta_{\Sigma\Xi} + \frac{g_{\text{YM}}^2}{\epsilon} Z_{\Sigma\Xi}^{(1)} \tag{2.10}$$

and thus the anomalous dimension matrix is

$$\gamma_{\Sigma\Xi} = 2 g_{\text{YM}}^2 \, Z_{\Sigma\Xi}^{(1)}\,. \tag{2.11}$$

[2]This notation will be shortly changed to a (hopefully) more convenient one.

[3]Of particular interest for us is the case when this is the set of all single- and double-trace operators with $\Delta - J = 2$ at tree level. This distinction is irrelevant at this point.

Thus, at this level, the entries of the anomalous dimension matrix can be determined in terms of the coefficient of the $1/\epsilon$ in 1-loop diagrams with one insertion of the composite operator.

As pointed out before, the relation between the matrix elements of the hamiltonian and the anomalous dimension matrix depends on the scalar product. More explicitly, assume that

$$\langle \bar{\mathcal{O}}_\Sigma(x)\mathcal{O}_\Xi(y)\rangle = S_{\Sigma\Xi}\frac{1}{|x-y|^{d_\Sigma+d_\Xi}}\,, \tag{2.12}$$

or schematically, as a scalar product,

$$\langle \mathcal{O}_\Sigma\mathcal{O}_\Xi\rangle = S_{\Sigma\Xi}\,. \tag{2.13}$$

Then, the matrix elements of $\Delta - J$ are given by

$$\langle \mathcal{O}_\Sigma(\Delta - J)\mathcal{O}_\Xi\rangle = \sum_\Psi \left(\gamma_{\Xi\Psi} + (d_\Xi - J_\Xi)\delta_{\Xi\Psi}\right) S_{\Sigma\Psi}\,. \tag{2.14}$$

The identification of $\Delta - J$ with the string hamiltonian P^-, requires that it be hermitean. This is reasonable since Δ is the "hamiltonian" of the euclidean theory using radial quantization. On the other hand, $\boldsymbol{\gamma}$ needs not be hermitean. The above equation resolves this concern about the interpretation of Δ, requiring only that $S\boldsymbol{\gamma}^T$ be hermitean.

As we will see in the following sections, for the operators we are interested in, the anomalous dimension matrix is *not* symmetric but the Hermiticity of the hamiltonian is recovered.

2.3 Divergences

Before we proceed with the details of the computation for operators dual to massive states generated by two creation operator let us describe the various ingredients that enter, not only in this computation, but also in the ones required for general operators.

The BMN limit is a large N limit which is different from the usual 't Hooft limit. In the 't Hooft limit perturbation theory is organized as a double series with expansion parameters $\lambda = g_{\rm YM}^2 N$ and $1/N$. The first one counts the loop order of the diagram while the latter counts the genus of the diagram. For large N high powers of $1/N$ can be safely neglected, and the dominant contribution to all processes comes from planar diagrams. Similarly, in the BMN limit there are two expansion parameters: $\lambda' = g_{\rm YM}^2 N/J^2$ and $g_2 = J^2/N$. As in the 't Hooft limit, the exponent of λ' counts the number of loops in the diagram. However, now g_2 counts the genus. In the limit in which the string theory is supposed to be described as a gauge theory this parameter is of order one, and thus higher genus diagrams cannot be neglected.

On the string theory side the perturbation theory is organized following the genus of the Feynman diagrams, with coefficients g_s^χ, where $\chi = 2g - 2$ is the Euler number of the diagram. Since g_2 expressed in string theory variables is proportional to g_s, it follows that higher genus gauge theory diagrams contain informations about string perturbation theory.

From this discussion it is clear why the mixing of single- and double-trace operators is parametrically relevant for the string splitting and joining. First, the mixing in question

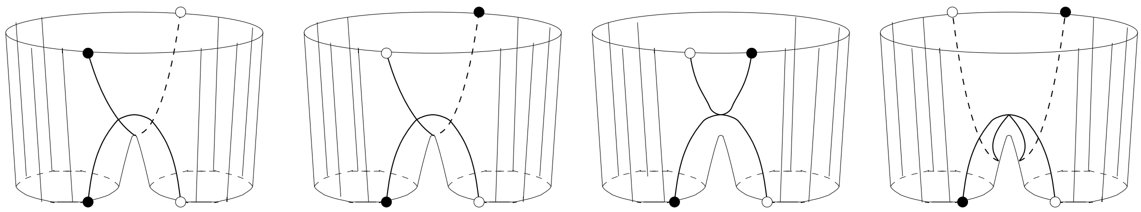

Figure 1: Feynman diagrams leading to the mixing of double-trace operators with single-trace operators, drawn on a pair of pants.

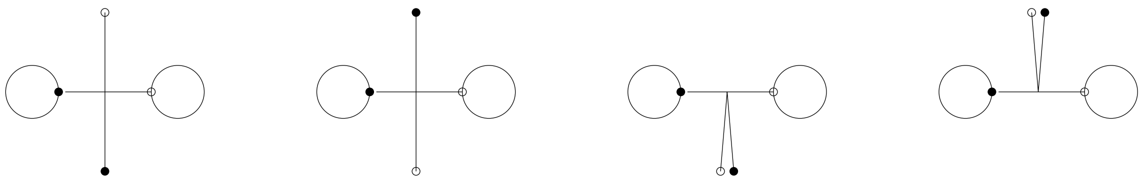

Figure 2: Symbolic representation of the diagrams in figure 1.

starts occurring at 1-loop; thus, it will be proportional to λ'. Then, diagrammatically, the relevant Feynman diagrams can be drawn on a sphere with three holes, a pair of pants. This counts as "genus 1/2" and thus the amplitude will be proportional to g_2. Expressing this product in string theory variables we find that $\lambda' g_2 = g_s$, which is the correct weight of the 3-string vertex.

We now describe in more detail the Feynman diagrams that will lead to the entries of the anomalous dimension matrix describing the mixing of single- and double-trace operators. Let us consider a double-trace operator and let us draw each of the traces composing it as a circle. Since we want to find the admixture of single-trace operators in the scale transformations of this operator, we are interested in the Feynman diagrams, with one insertion of a single trace operato, so that all external lines appear in the same trace. At 1-loop level only two fields can participate in the interaction. If both of them belong to the same trace, then such diagram will produce the anomalous dimension of the double-trace operator, and the mixing with single-trace ones is only due to the tree-level mixing of these operators. It is therefore clear that, to find the mixing due to interactions, the two interacting fields must belong to different traces of the original operator. All relevant Feynman diagrams with this property are drawn, on a sphere with three holes, in figure 1.

In these diagrams the fields taking part in the interactions are denoted by an empty and a filled circle; they can be either Z-fields or impurity fields. The spectator fields are denoted by straight lines and the top circle denotes the fact that outgoing fields are all under the same trace. These are all 1-loop diagrams since all fields in the double-trace operator are at the same space-time point. There are more 1-loop diagrams that take a double-trace operator and transform it in a single-trace one. Examples of such diagrams can be obtained by interchanging the black and the white dots on the top circles in the diagrams in figure 1. However, as we will see shortly, only the diagrams in figure 1 are divergent and thus they are the only ones that contribute to the anomalous dimension matrix.

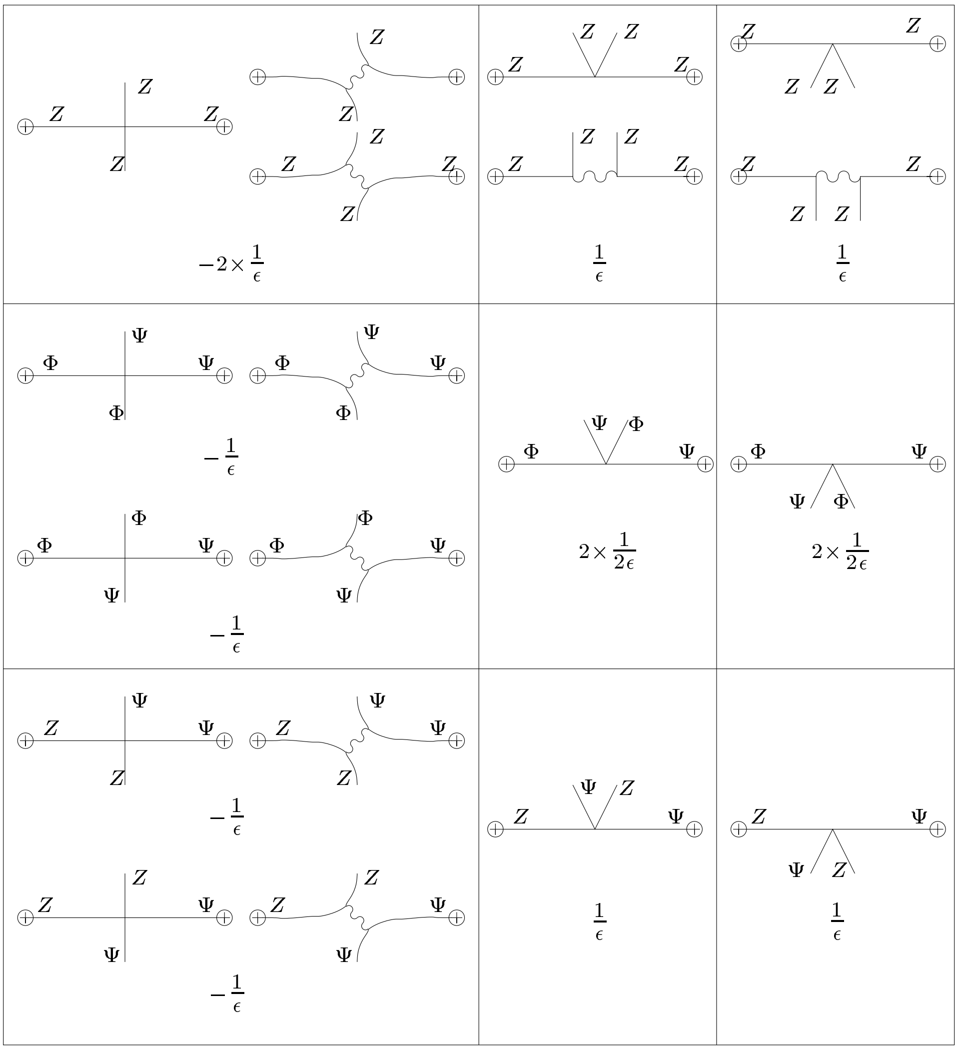

Figure 3: Divergent 1-loop diagrams.

Let us now introduce a convenient, schematic notation for these diagrams, which reduces the problem of computing them to just a combinatorial one. As above, we denote a trace by one circle and the fields that interact by empty and filled dots. We will omit the spectator fields as well as the fact that the outgoing fields are under the same trace. Then, we represent the four diagrams above as in figure 2. They are just projections of the diagrams in figure 1 on the plane containing the two circles denoting the double-trace operator. To turn this in a combinatorial problem we have to compute all the divergent 1-loop diagrams and then identify the black and white dots. The results of this computation are summarized in figure 3. The results stated in figure 3 include their combinatorial factors arising from the lagrangian. The points denoted by $\oplus$ can be either a black dot or a white dot. Thus, computing all diagrams represented in figure 1 reduces to finding all possible ways of linking the two traces building a double-trace operator using links appearing in figure 3.

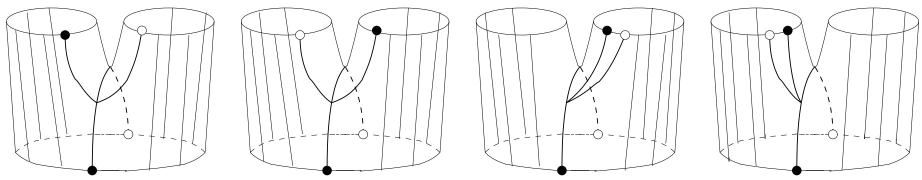

Figure 4: Feynman diagrams leading to the mixing of single-trace operators with double-trace operators, drawn on a pair of pants.

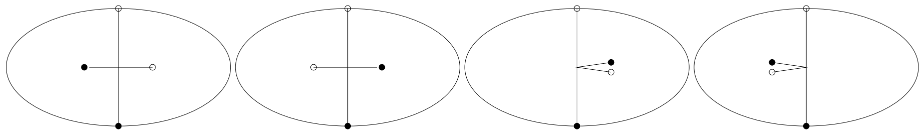

Figure 5: Symbolic representation of the diagrams in figure 4 .

A similar discussion holds for the diagrams leading to the mixing of single-trace with double-trace operators. We start with a single-trace operator, denoted by one circle, while the fields participating in the interaction are denoted by a black and a white dot. Turning it into a double-trace operator via an interaction vertex requires that the participating fields are not neighbors in the original operator. If they were, then the Feynman diagrams would yield the anomalous dimension of the single-trace operator and the mixing with the double-trace operators arises because of the tree-level mixing. For our purpose, the relevant Feynman diagrams, drawn on spheres with three holes, are listed in figure 4. The spectator fields are again represented by straight lines.

These diagrams can be also reduced to a combinatorial problem, as in the case of the mixing of double- with single-trace operators. With the same notation for the trace and for the interacting fields, the Feynman diagrams in figure 4 are shown in figure 5, respectively. As before, the spectator fields are not represented. Also, the resulting double-trace operator is not drawn explicitly, but it is fairly obvious from the figure. This representation is just a (simplified) projection on the plane containing the circle denoting the single-trace operator. As before, the problem of computing the Feynman diagrams yielding a double-trace operator out of a single-trace one is reduced to finding all ways of linking two non-neighboring fields in the single-trace operator using links from figure 3.

It is worth noting that not all the diagrams in figure 4 are not just the upside-down version of the diagrams in figure 1. In particular, we notice that inverting the last two diagrams in figure 1 leads to diagrams in which the mixing between single- and double-trace operators is due to their tree-level mixing and thus it must not be considered in the computation of the anomalous dimension matrix.

Before we continue with the main computation, let us demonstrate the use of the building blocks in figure 3, by rederiving the well-known [1, 2, 6], 1-loop anomalous dimension of the operator with a single impurity:

$$\mathcal{O} = \sum_{l=0}^{J} e^{il\varphi_m^J} Z^l \Phi Z^{J-l} \,. \tag{2.15}$$

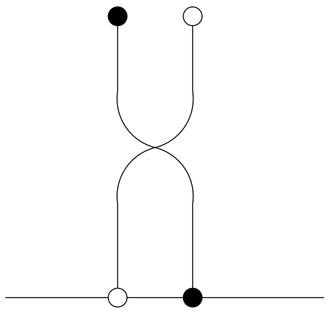

Figure 6: Feynman diagrams leading to the anomalous dimension of single-trace operators.

In the schematic notation of figure 2 and 5, the relevant diagrams are drawn in figure 6. The white and black dots can be either a Z field or the Φ field. The remaining diagrams, i.e. those that contribute to the wave function renormalization of the external lines, are not represented as they are not needed later on. We will use their expression from [2].

Using the second entry on the third line in figure 3 we find that part of the needed counterterm (up to terms arising for finite J) is:

$$\delta_1 \mathcal{O} = -\frac{1}{\epsilon}\left[e^{i\varphi_m^J}\mathcal{O} + e^{-i\varphi_m^J}\mathcal{O}\right]. \tag{2.16}$$

The first term arises by identifying the white dot with a Z-field, while the second one from identifying the white dot with Φ.

The second part of the counterterm arises by identifying both the black and the white dots with Z-fields and using the second entry on the first line in figure 3:

$$\delta_2 \mathcal{O} = -\frac{1}{\epsilon}(J-1)\mathcal{O}\,. \tag{2.17}$$

The third and last part comes from the wave function renormalization of the external lines. From [2] we have:

$$\delta_3 \mathcal{O} = \frac{1}{\epsilon}(J+1)\mathcal{O} \tag{2.18}$$

Combining everything, and restoring the coupling constant dependence, we find the following counterterm:

$$(\delta_1+\delta_2+\delta_3)\mathcal{O} = \frac{\lambda}{\epsilon}(2 - e^{i\varphi_m^J} - e^{-i\varphi_m^J})\mathcal{O} \tag{2.19}$$

which leads to the correct anomalous dimension for this operator:

$$\gamma_n = -2\lambda(e^{i\phi}+e^{-i\phi}-2) \simeq 4\lambda\phi^2 = 8\pi^2 n^2 \frac{\lambda}{J^2}\,. \tag{2.20}$$

To all loop orders and for an arbitrary number of impurities we found [2],[3], up to boundary contributions:

$$\gamma_{nn} = \sum_m N_m \gamma_m \qquad \text{where} \qquad \gamma_m = \sqrt{1 + 16\pi^2 m^2 \frac{g_{\mathrm{YM}}^2 N}{J^2}} - 1\,. \tag{2.21}$$

The preliminaries presented in this section can be used to compute the entries of the anomalous dimension matrix associated to any single- and multiple-trace operators. In the following we will concentrate on operators which are simple enough for a reasonably-sized exposition yet complex enough to exhibit most (hopefully all) possible subtleties.

3. Massive states generated by two creation operator; detailed analysis

In this section we discuss in detail the computation of the 1-loop anomalous dimension matrix for BMN operators with two impurities, i.e. operators that are dual to the simplest massive string states created by two creation operators acting on the vacuum. The relevant operators are:

$$
\begin{aligned}
\mathcal{O}_m^{[\Phi,\Psi]_J} &= K_m^{[\Phi,\Psi]_J} \sum_{l=0}^{J} e^{2\pi i l \varphi_m^J}\,\mathrm{Tr}[\Phi Z^l \Psi Z^{J-l}] \qquad \varphi_m^J = \frac{m}{J} \qquad m = 0,\dots,J-1 \\
\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}} &= K_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}\,\mathrm{Tr}[\Phi Z^{J_1}]\,\mathrm{Tr}[\Psi Z^{J-J_1}] \\
\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} &= K_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} \sum_{l=0}^{J_1} e^{2\pi i l \varphi_m^1}\,\mathrm{Tr}[\Phi Z^l \Psi Z^{J_1-l}]\,\mathrm{Tr}[Z^{J_2}]
\end{aligned} \tag{3.1}
$$

with the normalization constants K above are given by:

$$
\begin{aligned}
K_m^{[\Phi,\Psi]_J} &= \frac{1}{N^{(J+2)/2}\sqrt{J+1}} \qquad K_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}} = \frac{1}{N^{(J+2)/2}} \\
K_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} &= \frac{1}{N^{(J+2)/2}\sqrt{J-J_1}\sqrt{J_1+1}}\,.
\end{aligned} \tag{3.2}
$$

In equation (3.1) refer to any of the four real fields Φ_i and, for simplicity, we will take them to be distinct, $\Phi \neq \Psi$. To lowest order in the string coupling constant these operators correspond to:

$$
\begin{aligned}
\mathcal{O}_m^{[\Phi,\Psi]_J} &\to \alpha^{\dagger\Phi}_{-m}\alpha^{\dagger\Psi}_{m}|0\rangle \\
\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}} &\to \frac{1}{\sqrt{2}}\left(\alpha^{\dagger\Phi}_{0}|0\rangle \otimes \alpha^{\dagger\Psi}_{0}|0\rangle + \alpha^{\dagger\Psi}_{0}|0\rangle \otimes \alpha^{\dagger\Phi}_{0}|0\rangle\right) \\
\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} &\to \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \alpha^{\dagger\Phi}_{-m}\alpha^{\dagger\Psi}_{m}|0\rangle + \alpha^{\dagger\Phi}_{-m}\alpha^{\dagger\Psi}_{m}|0\rangle \otimes |0\rangle\right).
\end{aligned} \tag{3.3}
$$

Let us comment on the range of the index m in the first and last line in equation (3.1). There are $J+1$ independent operators of the type $\mathrm{Tr}[\Phi Z^l \Psi Z^{J-l}]$ and out of them we can build only $J+1$ independent operators: $\mathcal{O}_m^{[\Phi,\Psi]_J}$ with m taking values from 0 to $J-1$ and another operator which is orthogonal on $\mathcal{O}_m^{[\Phi,\Psi]_J}$ for all m and will be discarded. Later on, we will take the limit $N, J \to \infty$ and we will need to sum over the index m and it turns out that we have the sum from $-\infty$ to $+\infty$. The reason for this is the following. At finite N and J all functions we encounter are periodic under $m \to m+J$, a typical function is $\sin^2 \pi m/J$. However, once we expand for large J, we have access only to the increasing branch of the sine. To recover the missing piece we should take m to run from $-\infty = [J/2]$ to $+\infty = [J/2]$. A similar discussion applies to $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$, except that now m runs from $[J_1/2]$ to $[J_1/2]$.

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We will use the following indices for the anomalous diemnsion matrix and the various transformation matrices we will encounter later: a lower-case letter from the middle of the alphabet, say i, will correspond to a single-trace operator and an upper case letter from the beginning of the alphabet, say A, will denote a double-trace operator. The set of double-trace operators is naturally split in two subsets: the first subset contains only the operator $\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$ which will be denoted by an x index, and the operators $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ which will be denoted by a lower case greek index from the beginning of the alphabet, say α.

We will also use the following notation for the building blocks of BMN operators:

$$\tilde{O}_l^{[\Phi,\Psi]_J} = \mathrm{Tr}[\Phi Z^l \Psi Z^{J-l}] \tag{3.4}$$

in terms of which they are:

$$\mathcal{O}_m^{[\Phi,\Psi]_J} = K_m^{[\Phi,\Psi]_J} \sum_{l=0}^{J} e^{2\pi i n \frac{l}{J}} \tilde{O}_l^{[\Phi,\Psi]_J} \,. \tag{3.5}$$

The operators $\mathcal{O}$ are a kind of discrete Fourier transform of the operators $\tilde{O}$. However, the parallel is not exact at finite J, since there are $(J+1)$ independent operators $\tilde{O}$ and only J operators $\mathcal{O}$, since $\mathcal{O}_n^J = \mathcal{O}_{n+J}^J$.

To find the expression of $\tilde{O}_l^J$ in terms of BMN operators $\mathcal{O}_m^{[\Phi,\Psi]_J}$, we first notice that in $\mathcal{O}_m^{[\Phi,\Psi]_J}$ only the combination $(\tilde{O}_0^J + \tilde{O}_J^J)$ appears. Thus, for our purpose, we have J operators $\mathcal{O}$ as linear combinations of J operators $\tilde{O}$. The relations (3.5) can easily be inverted with the result:

$$\begin{aligned} K^{[\Phi,\Psi]_J} \tilde{O}_l^J &= \frac{1}{J} \sum_{n=0}^{J-1} e^{-2\pi i\, l \frac{n}{J}} \mathcal{O}_n^{[\Phi,\Psi]_J} \qquad (\forall) \quad l = 1 \ldots (J-1) \\ K^{[\Phi,\Psi]_J} (\tilde{O}_J^J + \tilde{O}_0^J) &= \frac{1}{J} \sum_{n=0}^{J-1} \mathcal{O}_n^{[\Phi,\Psi]_J} \,. \end{aligned} \tag{3.6}$$

However, at infinite J the situation is slightly different. In particular, we can take separately $K^{[\Phi,\Psi]_J} \tilde{O}_J^J$ and $K^{[\Phi,\Psi]_J} \tilde{O}_0^J$ to be equal to the right-hand-side of the second equation (3.6).

3.1 Order of limits

The issue to be addressed is then when is the limit $N \to \infty$ (which implies $J \to \infty$ according to the BMN limit) supposed to be taken? To answer this question let us go back to the single-trace operators $\mathcal{O}_m^{[\Phi,\Psi]_J}$, their scalar product and the boundary contributions to their anomalous dimensions. Let us first compute the scalar product at finite N and J. It is easy to see that it is:

$$\langle \bar{\mathcal{O}}_m^{[\Phi,\Psi]_J} \mathcal{O}_n^{[\Phi,\Psi]_J} \rangle = \delta_{mn} + \frac{1}{J} \,. \tag{3.7}$$

This non-orthogonality of order $1/J$ implies that $\mathcal{O}_n^{[\Phi,\Psi]_J}$ are not eigenvectors of the generator of scale transformations Δ at any order in λ' and g_2 if J and N are finite. This is the case because Δ is hermitean and the eigenvalues that are supposed to correspond to $\mathcal{O}_n^{[\Phi,\Psi]_J}$ are different for different n. A short computation reveals that the boundary contributions, i.e. terms due to the collision of Φ and Ψ, spoil the desired properties of the set of operators $\mathcal{O}_n^{[\Phi,\Psi]_J}$. Indeed, it turns out that

$$\begin{aligned}\Delta\mathcal{O}_m^{[\Phi,\Psi]_J} &= 4g_{\rm YM}^2N\left(4\sin^2\pi\varphi_m^J\right)\mathcal{O}_m^{[\Phi,\Psi]_J} - \\ &\quad - 2\frac{g_{\rm YM}^2N}{N^{(J+2)/2}\sqrt{J+1}}\left(4\sin^2\pi\varphi_m^J\right)\left(\tilde{O}_0^{[\Phi,\Psi]_J}+\tilde{O}_J^{[\Phi,\Psi]_J}\right) - \\ &\quad - 2\frac{g_{\rm YM}^2N}{N^{(J+2)/2}\sqrt{J+1}}\left(2i\sin 2\pi\varphi_m^J\right)\left(\tilde{O}_0^{[\Phi,\Psi]_J}-\tilde{O}_J^{[\Phi,\Psi]_J}\right). \end{aligned} \tag{3.8}$$

The appearance of the last term is troublesome since its coefficient is not finite as N and J are taken to infinity. This problem can be solved as follows. It is easy to match the construction of string states with the construction of representations of the symmetry algebra of the plane wave using creation operators acting on the light-cone vacuum [4]. This matching suggests that the number of BMN operators and the result of their transformations under symmetries of the plane wave matches the number of string states. All operators that are orthogonal to all the BMN operators can then be discarded. This is the case of the operator $(\tilde{O}_0^{[\Phi,\Psi]_J}-\tilde{O}_J^{[\Phi,\Psi]_J})$ appearing in the last line of (3.8). This orthogonality is the reason this term did not appear in the 2-point function computations of [6] and [9].

Thus, for our purpose, we can write:

$$\begin{aligned}\Delta\mathcal{O}_m^{[\Phi,\Psi]_J} &= 4g_{\rm YM}^2N\left(4\sin^2\pi\varphi_m^J\right)\mathcal{O}_m^{[\Phi,\Psi]_J} - \\ &\quad - 2\frac{g_{\rm YM}^2N}{N^{(J+2)/2}\sqrt{J+1}}\left(4\sin^2\pi\varphi_m^J\right)\left(\tilde{O}_0^{[\Phi,\Psi]_J}+\tilde{O}_J^{[\Phi,\Psi]_J}\right). \end{aligned} \tag{3.9}$$

This implies that, as stated before, $\mathcal{O}_m^{[\Phi,\Psi]_J}$ is not an eigenvector of the generator of scale transformations, the mismatch being of order $1/\sqrt{J}$ and therefore it does not have a definite anomalous dimension for finite N and J. At this stage it seems that the natural solution would be to add $(\tilde{O}_0^{[\Phi,\Psi]_J}+\tilde{O}_J^{[\Phi,\Psi]_J})$ to the set of operators $\mathcal{O}_m^{[\Phi,\Psi]_J}$ and construct the eigenvectors of Δ. The mixing would be of order $1/J$.

However, the correspondence between string theory in plane wave background and gauge theory is supposed to hold in the large N and J limits and this mixing would just make life complicated. In support of this last statement, we notice that the expectation value of Δ for finite N is identical to the one for infinite N:

$$\langle\mathcal{O}_m^{[\Phi,\Psi]_J}\Delta\mathcal{O}_m^{[\Phi,\Psi]_J}\rangle = 4g_{\rm YM}^2N\left(4\sin^2\pi\varphi_m^J\right). \tag{3.10}$$

This is because the $1/J$ term in the scalar product of operators $\mathcal{O}_m^{[\Phi,\Psi]_J}$ is canceled by the second term in the equation above.

This observation suggests that we can take the limit $N\to\infty$ at the beginning of the computations. This prescription solves the problems mentioned above. In particular, the BMN operators are orthonormal at tree level[4] and are the right gauge theory duals of the

[4]In computing the scalar product of operators, changing the summation over l from 0 to J to an integral, as in [6], is equivalent to taking $N\to\infty$ at the beginning of the computation.

states of the free string theory to all orders in λ' and zero-th order in g_2. The equation (3.6) can then be modified to a more symmetric form

$$K^{[\Phi,\Psi]_J}\tilde{O}_l^J = \frac{1}{J}\sum_{n=0}^{J-1} e^{-2\pi i\, l\frac{n}{J}}\mathcal{O}_n^{[\Phi,\Psi]_J} \qquad (\forall) \quad l = 0\ldots J(=\infty)\,. \tag{3.11}$$

These subtleties can be avoided by slightly changing the definition of the phase appearing in equation (3.1) to:

$$\varphi_m^J = \frac{m}{J+1}\,. \tag{3.12}$$

At order $\mathcal{O}(g_2^0)$ these operators are eigenvectors of the generators of scale transformations to all orders in λ' and their scalar product becomes diagonal to all orders in $1/J$. The expression of $\tilde{O}_l^J$ in terms of these modified operators is obtained from (3.11) via obvious replacements. These modifications were also suggested in [20].

An alternative way of performing the computations, which does not require writting $\tilde{O}$ in terms of $\mathcal{O}$, goes as follows. We first compute the contribution to the matrix elements of the hamiltonian due to the mixing of the double-trace and single-trace operators induced by the interactions described in the previous section. This is the matrix:

$$H = S\boldsymbol{\gamma}^T\,. \tag{3.13}$$

Then, we perform the change of basis that diagonalizes the scalar product matrix, S:

$$S \longrightarrow \hat{S} = USU^T = \mathbb{1}\,. \tag{3.14}$$

Under this transformation, the hamiltonian matrix transforms as:

$$H \longrightarrow \hat{H} = US\boldsymbol{\gamma}^T U^T\,. \tag{3.15}$$

Since the new basis is orthonormal, $\hat{H}$ is the appropriate matrix for comparison with string theory.

In the following sections we will adopt the first computational scheme, i.e. we will compute the anomalous dimension matrix in the limit $N \to \infty$. We will then proceed in the next section with the necessary change of basis.

3.2 Scale transformations of double-trace operators

Having discussed in detail all necessary ingredients, let us now proceed with the computation of the entries of the anomalous dimension matrix describing the appearence of single-trace operators in the scale transformations of double-trace ones. Given the qualitative differences between $\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$ and $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$, we will treat then separately. For both cases the template Feynman diagrams are those in figure 2 with the empty and filled dots being either a Z or an impurity.

3.2.1 $\mathcal{O}_{00}^{[\Phi]_{J_1}[\Psi]_{J-J_1}} \longrightarrow \mathcal{O}_m^{[\Phi,\Psi]_J}$

• $Z - Z$ interactions:

Each of the diagrams in figure 2 gives a double sum, since each of the two Z-s can sit anywhere in the original string of Z-s of each operator. The first two diagrams are, of course, identical. Taking into account the relative factors in the first line of figure 3 we find that the relevant combination of operators is:

$$\sum_{p=0}^{J_2-1}\sum_{q=0}^{J_1-1}\Big[2\,\mathrm{Tr}[\Psi Z^{p+q+1}\Phi Z^{J-p-q-1}] - \mathrm{Tr}[\Psi Z^{p+q+2}\Phi Z^{J-p-q-2}] - \mathrm{Tr}[\Psi Z^{p+q}\Phi Z^{J-p-q-1}]\Big]\,. \tag{3.16}$$

It is clear that most terms cancel and we are left with a counterterm equal to:

$$\mathbb{C}_1 = \frac{1}{\epsilon} g_{\mathrm{YM}}^2 \mathbb{T}(\Psi,\Phi)\,, \tag{3.17}$$

where $\mathbb{T}(\Psi,\Phi)$ is defined to be the result of the sum in (3.16):

$$\mathbb{T}(\Psi,\Phi) = \mathrm{Tr}[\Psi Z^{J_1}\Phi Z^{J_2}] + \mathrm{Tr}[\Psi Z^{J_2}\Phi Z^{J_1}] - \mathrm{Tr}[\Psi\Phi Z^{J_1+J_2}] - \mathrm{Tr}[\Phi\Psi Z^{J_1+J_2}]\,. \tag{3.18}$$

• impurity – impurity interactions:

In this case each diagram in figure 2 gives exactly one term. Taking into account the second line of figure 3, the required counterterm is:

$$\mathbb{C}_2 = \frac{1}{\epsilon} g_{\mathrm{YM}}^2 \mathbb{T}(\Psi,\Phi)\,. \tag{3.19}$$

• Z – impurity interactions:

Interactions ΦZ and ΨZ are described by the third line in figure 3 as well as six more diagrams in which Ψ is replaced by Φ and its position is switched with the position of Z.

Each diagram in figure 2 leads to a simple sum, corresponding to the position of the Z field in the original oprator. It is not hard to see that these sums cancel pairwise. Indeed, consider the case in which the filled dot in figure 2 represents an impurity, say Φ, while the empty one is a Z field. Then, taking into account the relative signs in the third line of figure 3 we have the following coefficient for $\frac{1}{\epsilon}$:

$$\sum_{p=0}^{J_2-1}\mathrm{Tr}[\Psi Z^{J_1+p+1}\Phi Z^{J_2-p-1}] + \sum_{p=0}^{J_1-1}\mathrm{Tr}[\Psi Z^{p}\Phi Z^{J-p}] - \sum_{p=0}^{J_2-1}\mathrm{Tr}[\Psi Z^{J_1+p+1}\Phi Z^{J_2-p-1}] - \sum_{p=0}^{J_1-1}\mathrm{Tr}[\Psi Z^{p}\Phi Z^{J-p}] = 0 \tag{3.20}$$

and they obviously cancel. Each term in the equation above is associated to one of the four diagrams in figure 2, respectively.

Thus, the total counterterm needed for the renormalization of $\mathcal{O}_x$ is:

$$\mathbb{C} = \frac{2}{\epsilon} g_{\mathrm{YM}}^2 \mathbb{T}(\Phi,\Psi)\,. \tag{3.21}$$

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Using equations (3.6) we can express $\mathbb{T}$ in terms of BMN operators:

$$\mathbb{T} = -\frac{4}{\sqrt{J}} \sum_{m=0}^{J-1} \sin^2\left(\pi\, m \frac{J_1}{J}\right) \mathcal{O}_m^{[\Phi,\Psi]_J} . \tag{3.22}$$

Thus, the total renormalization constant for the mixing between a double-trace operator built out of two BPS states and the single-trace BMN operators becomes:

$$Z_x{}^i = -\frac{1}{\epsilon} \frac{8g_{\rm YM}^2}{\sqrt{J}} \sin^2\left(\pi\, m_i \frac{J_i}{J}\right) \qquad (\forall) \qquad m_i = 0, \ldots, J-1 . \tag{3.23}$$

Using (2.11), this immediately leads to the following entries of the anomalous dimension matrix:

$$\gamma_{x;i} = -\frac{16g_{\rm YM}^2}{\sqrt{J}} \sin^2\left(\pi\, m_i \frac{J_2}{J}\right), \tag{3.24}$$

where in the last equality we used the coefficient of the free 3-point function derived in the appendix.

3.2.2 $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} \longrightarrow \mathcal{O}_m^{[\Phi,\Psi]_J}$

Now we turn to the computation of the admixture of single-trace operators in the scale transformation of the double-trace operators $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$. Since one of the operators building $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ does not contain any impurities, there are only two types of interactions to analyze: $Z - Z$ and $Z -$ impurity.

However, things are somewhat simpler than they seem, since it is easy to see that the $Z - Z$ interactions vanish. Indeed, let us consider more general operators, of the form $\mathrm{Tr}[AZ^p]$ and $\mathrm{Tr}[BZ^q]$. Particular choices of A and B lead to the operators $\tilde{O}_l$. Using the first line in figure 3, we find that the divergence in a 1-loop graph with one insertion of the product of these operators is proportional to:

$$\mathrm{Tr}[AZ^pBZ^q] + \mathrm{Tr}[AZ^qBZ^p] - \mathrm{Tr}[ABZ^{p+q}] - \mathrm{Tr}[BAZ^{p+q}] . \tag{3.25}$$

For the operators $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ one of the operators A and B is the identity operator and thus, the sum above vanishes.

Thus, unlike the case of operator $\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$, now the Z-impurity interactions give the nonvanishing contribution. Taking into account the normalization factors, each of the $J+1$ terms of $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ gives the following counterterms:

$$\mathbb{C}_l = \frac{1}{\epsilon} \frac{1}{\sqrt{J_1 J_2}} 2J_2\, \S_l \tag{3.26}$$

with $\S_l$ defined as

$$\begin{aligned} \S_l \;=\; & \mathrm{Tr}[\Phi Z^l \Psi Z^{J-l}] - \mathrm{Tr}[\Phi Z^{l+1} \Psi Z^{J-l-1}] + \\ & + \mathrm{Tr}[\Phi Z^{J_2+l} \Psi Z^{J_1-l}] - \mathrm{Tr}[\Phi Z^{J_2+l-1} \Psi Z^{J_1-l+1}] . \end{aligned} \tag{3.27}$$

Each of the four terms above comes from one of the four diagrans in figure 2. The extra factor of 2 in (3.26) is due to the fact that both Φ and Ψ bring the same contribution while the extra factor of J_2 is due to the J_2 equivalent choices of Z in the $\mathrm{Tr}[Z^{J_2}]$ factor of $\mathcal{O}_{m,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$.

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Let us consider first the case $n = 0$, i.e. $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ is a product of BPS operators. Then the sum over l simplifies considerably and we find

$$\sum_{l=0}^{J_1} \S_l = -\,\mathrm{Tr}[\Phi Z^{J_1+1}\Psi Z^{J_2-1} + \Phi Z^{J_2-1}\Psi Z^{J_1+1} - \Phi\Psi Z^J - \Psi\Phi Z^J]\,. \tag{3.28}$$

Using equation (3.6) the required counterterms can be expressed in terms of BMN operators as:

$$\mathbb{C} = \frac{8}{\epsilon}\sqrt{\frac{J_2}{J_1 J}}\sum_{m=0}^{J-1}\sin^2\left(\pi m\frac{J_1+1}{J}\right)\mathcal{O}_m^{[\Phi,\Psi]_J}\,. \tag{3.29}$$

Therefore, the corresponding entry of the anomalous dimension matrix is:

$$\gamma_{\alpha=0;i} = 16 g_{\mathrm{YM}}^2\sqrt{\frac{J_2}{J_1 J}}\sin^2\left(\pi m\frac{J_1+1}{J}\right), \tag{3.30}$$

where in the last equality we neglected terms suppressed by $1/J$ and we used the 3-point function coefficient from the appendix.[5]

We now turn to the case of general $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$. The sum over $\S_l$ is now:

$$\begin{aligned}\sum_{l=0}^{J_1} e^{il\varphi_n^{J_1}}\S_l = & \sum_{l=0}^{J_1}\S_l + \\ & + \sum_{t=0}^{J_1-1} e^{-it\varphi_n^{J_1}}\left(1-e^{-i\varphi_n^{J_1}}\right)\mathrm{Tr}[\Phi Z^{J_1-t}\Psi Z^{J_2+t}] + \\ & + \sum_{t=0}^{J_1-1} e^{it\varphi_n^{J_1}}\left(1-e^{i\varphi_n^{J_1}}\right)\mathrm{Tr}[\Phi Z^{J_2+t}\Psi Z^{J_1-t}],\end{aligned} \tag{3.31}$$

where $\varphi_n^{J_1} = 2\pi\frac{n}{J_1}$ and we have isolated the terms that do not vanish in the limit $n = 0$. Using equation (3.6) the last two sums can be expressed in terms of BMN operators; using that $exp(iJ_1\varphi_n^{J_1}) = 1$, the result is:

$$\frac{1}{\sqrt{J}}\sum_{m=0}^{J-1}\left[\left(e^{-iJ_1\varphi_m^J}-1\right)\frac{1-e^{-i\varphi_n^{J_1}}}{1-e^{-i(\varphi_n^{J_1}-\varphi_m^J)}} + \left(e^{iJ_1\varphi_m^J}-1\right)\frac{1-e^{i\varphi_n^{J_1}}}{1-e^{i(\varphi_n^{J_1}-\varphi_m^J)}}\right]\mathcal{O}_m^{[\Phi,\Psi]_J}\,. \tag{3.32}$$

To leading order in $1/J$ and $1/J_1$ this equation simplifies to:

$$\frac{1}{\sqrt{J}}\sum_{m=0}^{J-1}\frac{n}{m\frac{J_1}{J}-n}\,4\sin^2\left(\pi m\frac{J_1}{J}\right)\mathcal{O}_m^J \qquad x = \frac{J_1}{J}\,. \tag{3.33}$$

Therefore, the counterterms required for the renormalization of diagrams with one insertion of $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ are

$$\mathbb{C}_{full} = \frac{8g_{\mathrm{YM}}^2}{\epsilon}\sqrt{\frac{J_2}{J\,J_1}}\sum_{m=0}^{J-1}\frac{mx}{mx-n}\sin^2\left(\pi m\frac{J_1}{J}\right)\mathcal{O}_m^{[\Phi,\Psi]_J} \tag{3.34}$$

[5] In [6] this subleading term never appears because the sum over l is replaced by an integral. The corrections to this replacement are of order $1/J$

which in turn imply that the relevant entries in the anomalous dimension matrix are:

$$\gamma_{i,\alpha} = 16 g_{\rm YM}^2 \sqrt{\frac{J_2}{J\, J_1}} \frac{m_i x}{m_i x - n_\alpha} \sin^2\left(\pi m_i \frac{J_1}{J}\right), \tag{3.35}$$

where again we have used the 3-point function coefficients from the appendix.

3.3 Scale transformations of single-trace operators

To complete the computation of the entries of the anomalous dimension matrix describing the mixing of $\mathcal{O}_m^{[\Phi,\Psi]_J}$ and $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ under scale transformations we need to find $\gamma_{A\,i}$. The template Feynman diagrans are those that lead to the comultiplication (figure 5). As in the previous subsection, we will discuss $\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$ and $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$ separately.

Before we proceed let us stress two points. First, we will be working with an SU(N) theory. This will allow us to set to zero terms containing traces of a single field which might not be in line with the general formulae we will derive. Second, the contribution of the nearest neighbour interaction leads to the diagonal entries of the anomalous dimension matrix and was already discussed in our conventions at the end of section 2.3. We will however keep those terms as a check for the derivation of more general expressions.

3.3.1 $\mathcal{O}_m^{[\Phi,\Psi]_J} \longrightarrow \mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$

The mixing turns out to vanish. Indeed, the various interactions generate the following counterterms:

- ZZ :

$$\mathbb{C}_{ZZ} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}} \sum_{l=1}^{J-1} e^{il\varphi_m^J} \left(O_\Phi^l O_\Psi^{J-l} + O_\Phi^{J-l} O_\Psi^l\right) \tag{3.36}$$

- $\Phi\Psi$:

$$\mathbb{C}_{\Phi\Psi} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}} \sum_{l=1}^{J-1} e^{il\varphi_m^J} \left(O_\Phi^l O_\Psi^{J-l} + O_\Phi^{J-l} O_\Psi^l\right) \tag{3.37}$$

- $Z\Phi$:

$$\mathbb{C}_{Z\Phi} = \frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}} \sum_{l=1}^{J-1} e^{il\varphi_m^J} \left(O_\Phi^l O_\Psi^{J-l} + O_\Phi^{J-l} O_\Psi^l\right) \tag{3.38}$$

The $Z\Psi$ interactions produce the same result.

Thus, doubling the $Z\Phi$ and adding to it the ZZ and $\Phi\Psi$ contributions we find a vanishing counterterm, as stated above. Therefore,

$$\gamma_{i,x} = 0\,. \tag{3.39}$$

3.3.2 $\mathcal{O}_m^{[\Phi,\Psi]_J} \longrightarrow \mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$

For operators of this type the mixing turns out to be generically nonvanishing. Furthermore, all types of interactions give nonvanishing contributions. As in the previous subsections, most of our derivations are valid at finite J as well. A byproduct of this computation will be that at subleading order in $\frac{1}{J}$ the BMN operators mix with non-BMN ones.

Let us now analyze the three possible types of interactions:

- $Z-Z$

The required counterterm is:

$$\begin{aligned}\mathbb{C}_{ZZ} &= -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\times \\ &\times\left[\sum_{l=2}^{J} e^{il\varphi_m^J}\sum_{p=0}^{l-2}(l-p-1)\left[-2O^{p+1}\tilde{O}_{l-p-1}^{J-p-1}+O^p\tilde{O}_{l-p}^{J-p}+O^{p+2}\tilde{O}_{l-p-2}^{J-p-2}\right]+\right. \\ &\left.+\sum_{l=0}^{J-2} e^{il\varphi_m^J}\sum_{p=0}^{l-2}(l-p-1)\left[-2O^{p+1}\tilde{O}_{l}^{J-p-1}+O^p\tilde{O}_{l}^{J-p}+O^{p+2}\tilde{O}_{l}^{J-p-2}\right]\right],\end{aligned} \tag{3.40}$$

where $O^p=\mathrm{Tr}[Z^p]$. One of the sums can be done explicitly, with the result:

$$\begin{aligned}\mathbb{C}_{ZZ} &= -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\sum_{l=2}^{J}\left[e^{il\varphi_m^J}\tilde{O}_0^{J-l}+e^{-il\varphi_m^J}\tilde{O}_{J-l}^{J-l}\right]O^l- \\ &-\frac{1}{\epsilon}\frac{g_{\rm YM}^2 N}{\sqrt{J}}\left[(J+2)-4\right]\sum_{l=2}^{J} e^{il\varphi_m^J}\tilde{O}_l^J\,.\end{aligned} \tag{3.41}$$

The second line in the formula above is due to the nearest neighbour interaction and is one of the terms leading to the anomalous dimension of $\mathcal{O}_m^{[\Phi,\Psi]_J}$. As stated in the beginning, we kept it in place as a check of our computation and it is reassuring that its coefficient is the correct one: $J+2$ is canceled by the individual wave function renormalization of fields while the remaining (-4) is twice as large as the $Z-Z$ contribution to equation (2.20).

- $\Phi-\Psi$

There are two diagrams that give nonvanishing contributions to the mixing between $\mathcal{O}_m^{[\Phi,\Psi]_J}$ and $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$. They correspond to the second and third columns of the second line of figure 3. It is not hard to see that the required counterterm is:

$$\mathbb{C}_{\Phi\Psi} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\sum_{l=0}^{J}\left[e^{il\varphi_m^J}\tilde{O}_{J-l}^{J-l}+e^{-il\varphi_m^J}\tilde{O}_0^{J-l}\right]O^l\,. \tag{3.42}$$

- $Z\Phi$ and $Z\Psi$

As one probably expects, each type of interaction requires the same counterterm. Carefully following the diagrams in figure 5, the counterterms turn out to be:

$$\begin{aligned}\mathbb{C}_{Z\Phi} &= -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\left[\sum_{l=1}^{J} e^{il\varphi_m^J}\sum_{p=0}^{l-1}\left[O^p\tilde{O}_{l-p-1}^{J-p}-O^{p+1}\tilde{O}_{l-p-1}^{J-p-1}\right]+\right. \\ &\left.+\sum_{l=0}^{J-1} e^{il\varphi_m^J}\sum_{p=0}^{J-l-1}\left[O^p\tilde{O}_{l-p-1}^{l+1}-O^{p+1}\tilde{O}_{l}^{J-p-1}\right]\right].\end{aligned} \tag{3.43}$$

This equation can be simplified to:

$$
\begin{aligned}
\mathbb{C}_{Z-\Phi} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\Bigg\{ & \sum_{t=0}^{J-1}\Bigg[e^{-it\varphi_m^J}\sum_{p=1}^{J-t-1} O^p\left[\tilde{O}^{J-p}_{J-t-p-1}-\tilde{O}^{J-p}_{J-t-p}\right]+ \\
& + e^{it\varphi_m^J}\sum_{p=1}^{J-t-1} O^p\left[\tilde{O}^{J-p}_{t+1}-\tilde{O}^{J-p}_{t}\right]\Bigg]- \\
& - \sum_{l=1}^{J} O^l\left[e^{il\varphi_m^J}\tilde{O}^{J-l}_0+e^{-il\varphi_m^J}\tilde{O}^{J-l}_{J-l}\right]+ \\
& + N\sum_{l=1}^{J} e^{il\varphi_m^J}\tilde{O}^J_{l-1}+N\sum_{l=0}^{J-1} e^{il\varphi_m^J}\tilde{O}^J_{l+1}\Bigg\}\,. \qquad (3.44)
\end{aligned}
$$

As in the case of ZZ interactions, the terms on the last line are the source of anomalous dimension for the operator $\mathcal{O}_m^{[\Phi,\Psi]_J}$. Taking into account the contribution of both Φ and Ψ (i.e. doubling (3.44)) we indeed find the correct coefficient (equations (2.20) and (2.21)).

To extract the desired entries of the anomalous dimension matrix we isolate double-trace operators containing $O^{J_2}=\mathrm{Tr}[Z^{J-J_1}]$. The formulae simplify considerably:

$$
\mathbb{C}_{ZZ}+\mathbb{C}_{\Phi\Psi} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\left(e^{iJ_1\varphi_m^J}+e^{-iJ_1\varphi_m^J}\right)\left[\tilde{O}^{J_1}_0+\tilde{O}^{J_1}_{J_1}\right]O^{J_2}\,. \qquad (3.45)
$$

When isolating O^{J_2} from $Z_{Z\Phi}$ we have to be careful with the upper limit for the summation over t, since we want $1\le p=J_2\le J-t-1$. Taking this properly into account, the result is:

$$
\begin{aligned}
\mathbb{C}_{Z\Phi} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\Bigg\{ & \sum_{t=1}^{J_1-1}\left[e^{-it\varphi_m^J}(e^{i\varphi_m^J}-1)\tilde{O}^{J_1}_{J_1-t}+e^{it\varphi_m^J}(e^{-i\varphi_m^J}-1)\tilde{O}^{J_1}_{t}\right]O^{J_2}+ \\
& + e^{-it\varphi_m^J}(e^{i\varphi_m^J}-1)\tilde{O}^{J_1}_0O^{J_2}+e^{it\varphi_m^J}(e^{-i\varphi_m^J}-1)\tilde{O}^{J_1}_{J_1}O^{J_2}- \\
& - \left[\tilde{O}^{J_1}_0+\tilde{O}^{J_1}_{J_1}\right]O^{J_2}\Bigg\} \qquad (3.46)
\end{aligned}
$$

Combining everything we find that the counterterm required for the diagrams with one $\mathcal{O}_i$ insertion which produce a double-trace operator containing O^{J_2} is:

$$
\begin{aligned}
\mathbb{C}^{\rm split} = -\frac{1}{\epsilon}\frac{g_{\rm YM}^2}{\sqrt{J}}\Bigg\{ & 2\sum_{t=1}^{J_1-1}\left[e^{-it\varphi_m^J}(e^{i\varphi_m^J}-1)\tilde{O}^{J_1}_{J_1-t}+e^{it\varphi_m^J}(e^{-i\varphi_m^J}-1)\tilde{O}^{J_1}_{t}\right]O^{J_2}+ \\
& + 2\left[e^{-it\varphi_m^J}(e^{i\varphi_m^J}-1)\tilde{O}^{J_1}_0+e^{it\varphi_m^J}(e^{-i\varphi_m^J}-1)\tilde{O}^{J_1}_{J_1}\right]O^{J_2}+ \\
& + \left(e^{iJ_1\varphi_m^J}+e^{-iJ_1\varphi_m^J}-2\right)\left[\tilde{O}^{J_1}_0+\tilde{O}^{J_1}_{J_1}\right]O^{J_2}\Bigg\} \qquad (3.47)
\end{aligned}
$$

It is now a simple exercise to use equation (3.6) and compute the sum over t we find to express the $\mathbb{C}^{\text{split}}$ in terms of BMN operators:

$$
\begin{aligned}
\mathbb{C}^{\text{split}} &= -\frac{1}{\epsilon}\frac{g_{\text{YM}}^2}{\sqrt{J\,J_1}}\sum_{n=0}^{J_1-1}\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}\times\\
&\quad\times\Bigg\{-8\sin^2\left(\frac{1}{2}J_1\varphi_m^J\right)+2\Bigg[(1-e^{-iJ_1\varphi_m^J})\frac{e^{i\varphi_m^J}-1}{e^{i(\varphi_m^J-\varphi_n^{J_1})}-1}+\\
&\qquad\qquad\qquad\qquad+(1-e^{iJ_1\varphi_m^J})\frac{e^{-i\varphi_m^J}-1}{e^{-i(\varphi_m^J-\varphi_n^{J_1})}-1}\Bigg]\Bigg\}-\\
&\quad-\frac{1}{\epsilon}\frac{g_{\text{YM}}^2}{\sqrt{J}}\left\{e^{-it\varphi_m^J}(e^{i\varphi_m^J}-1)-e^{it\varphi_m^J}(e^{-i\varphi_m^J}-1)\right\}\left[\tilde{O}_0^{J_1}-\tilde{O}_{J_1}^{J_1}\right]O^{J_2}\,.
\end{aligned}
$$

Expanding this for large J_i we find:

$$
\mathbb{C}^{\text{split}} = -\frac{8g_{\text{YM}}^2}{\epsilon}\sqrt{\frac{J_2}{J\,J_1}}\sum_{m=0}^{J-1}\frac{n}{mx-n}\sin^2\left(\pi m\frac{J_1}{J}\right)\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}} \tag{3.48}
$$

which clearly leads to the following entries in the anomalous dimension matrix:

$$
\gamma_{\alpha,i} = -16g_{\text{YM}}^2\sqrt{\frac{J_2}{J\,J_1}}\frac{n}{mx-n}\sin^2\left(\pi m\frac{J_1}{J}\right). \tag{3.49}
$$

3.4 Symmetry

An essential test for identification of the hamiltonian is its Hermiticity. Since the scalar product is not diagonal we use equation (2.14) to calculate its matrix elements:

$$
H_{iA} = S_{i\Sigma}\gamma_{A\Sigma} = \gamma_{Ai} + C_{iA}\delta_A \qquad H_{Ai} = S_{A\Sigma}\gamma_{i\Sigma} = \gamma_{iA} + C_{iA}\delta_i\,. \tag{3.50}
$$

Thus we find that the hamiltonian is hermitean if

$$
\gamma_{Ai} - \gamma_{iA} = C_{iA}(\delta_i - \delta_A)\,. \tag{3.51}
$$

It is not hard to check that the anomaloud dimension matrix satisfies this requirement. To summarize the results of this long computation, we list here all entries of the anomalous dimension matrix:

$$
\begin{aligned}
\boldsymbol{\gamma} &\equiv \begin{pmatrix} \gamma_{ij} & \gamma_{iA} \\ \gamma_{Bj} & \gamma_{BA} \end{pmatrix} \qquad\qquad (3.52)\\
&= 16\pi^2\lambda'\begin{pmatrix} m_i^2\delta_{ij} & 0 & -\frac{g_2}{\pi^2\sqrt{J}}\sqrt{\frac{1-x}{x}}\frac{n_\alpha}{m_ix-n_\alpha}\sin^2\pi xm_i \\ -\frac{g_2}{\pi^2\sqrt{J}}\sin^2\pi xm_i & 0 & 0 \\ \frac{g_2}{\pi^2\sqrt{J}}\sqrt{\frac{1-x}{x}}\frac{m_ix}{m_ix-n_\alpha}\sin^2\pi xm_i & 0 & n_\alpha\delta_{\beta\alpha} \end{pmatrix}.
\end{aligned}
$$

Inspecting (3.53) and using the explicit expressions for δ and C it is easy to see that this equation is satisfied. Thus, at this level, the generator of scale transformations is hermitean and its interpretation as the string hamiltonian the interpretation is reasonable. This can be also interpreted as a test of the correctness of our computation.

4. Diagonalization of the scalar product

The scalar product of the states should be invariant under evolution, which is equivalent to the hamiltonian being symmetric. From the CFT point of view, this implies that in the basis where the matrix of the scalar products is the unit matrix, the matrix of the anomalous dimensions should be symmetric.

To extract the hamiltonian in a basis suitable for comparison with string theory we must diagonalize the scalar product of states. On the string theory side the 2-string states belong to a space which is the tensor product of 2 one-string states. This space is orthogonal to all orders in string coupling to the space of 1-string states; the transition from one to the other is given by the hamiltonian.

4.1 Order g_2; three-string vertex

As discussed in section 2, the gauge theory version of the string scalar product is defined by the 2-point function. Using the results in the appendix, we have the following matrix representation:

$$S \equiv \begin{pmatrix} S_{ij} & S_{iA} \\ S_{Bj} & S_{BA} \end{pmatrix} = \begin{pmatrix} \delta_{ij} & C_{iA} \\ C_{jB} & \delta_{BA} \end{pmatrix} \equiv \begin{pmatrix} \mathbb{1} & g_2 c \\ g_2 c^T & \mathbb{1} \end{pmatrix}, \tag{4.1}$$

where for later convenience we exibited the g_2 dependence of C. This equation is correct to leading order in $g_2 \sim g_s$. The diagonal entries are corrected at order g_2^2 [6, 9], while the scalar product between single- and double-trace operators is corrected at order g_2^3. Taking into account all corrections of order g_2^2 is the subject of the next subsection.

Let us now find a basis in which the scalar product is diagonal and the matrix elements of the generator of scale transformations match with the string theory predictions for the matrix elements of P^-. To this end let us introduce the operators $\hat{\mathcal{O}}$ related to $\mathcal{O}$ by some transformation matrix U:

$$\hat{\mathcal{O}}_\Sigma = \sum_\Pi U_{\Sigma\Pi} \mathcal{O}_\Pi \qquad \text{with} \qquad U = \begin{pmatrix} \delta_{ij} & g_2 a_{iA} \\ g_2 b_{Bj} & \delta_{AB} \end{pmatrix}, \tag{4.2}$$

where for later convenience we exhibited the g_2 of various entries. In this basis, the scalar product matrix is:

$$\hat{S} = USU^T \tag{4.3}$$

which equals to the identity matrix up to corrections of order g_2^2 provided that

$$c + a + b^T = 0\,. \tag{4.4}$$

Performing the transformation (4.2) on equation (2.8) we find that the anomalous dimension matrix expressed in the basis of operators $\hat{O}$ is related to γ by:

$$\hat{\boldsymbol{\gamma}} = U\boldsymbol{\gamma}U^{-1}\,. \tag{4.5}$$

It is clear that while γ is not symmetric, $\hat{\gamma}$ can be since U need not be an orthogonal matrix. Introducing the following notation for γ

$$\boldsymbol{\gamma} = \begin{pmatrix} \Delta_1 & g_2\Delta \\ g_2\Delta' & \Delta_2 \end{pmatrix} \tag{4.6}$$

we find that, up to corrections of order g_2^2, $\hat{\gamma}$ is given by:

$$\hat{\gamma} = \begin{pmatrix} \Delta_1 & g_2\left[\Delta - (\Delta_1 a - a\Delta_2)\right] \\ g_2\left[\Delta' + (b\Delta_1 - \Delta_2 b)\right] & \Delta_2 \end{pmatrix}. \tag{4.7}$$

It is easy to see that this matrix is symmetric provided the scalar product is the identity matrix. Indeed,

$$\begin{aligned} &[\Delta - (\Delta_1 a - a\Delta_2)] - [\Delta' + (b\Delta_1 - \Delta_2 b)] = \\ &\quad = \Delta - \Delta' - [\Delta_1(a + b^T) - \Delta_2(a + b^T)] = \Delta - \Delta' + [\Delta_1 c - \Delta_2 c] \end{aligned} \tag{4.8}$$

which is satisfied due to (3.51).

To find an explicit form for $\hat{\boldsymbol{\gamma}}$ we must solve (4.4). Clearly there is no unique solution. However, it is possible to show that all solutions to this equation are related by left-multiplication with a matrix that is orthogonal to order $O(g_2^2)$. To this end let us consider some fixed solution a_0 and b_0 to the equation above. Any other solution is related to this one by

$$a_0 \to a = a_0 + O_{12} \qquad b_0 \to b = b_0 - O_{12}^T\,. \tag{4.9}$$

This follows trivially by subtracting equation (4.4) written for a and b from the one written for a_0 and b_0.

Consider now an orthogonal matrix O

$$O = \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \quad O_{11}O_{11}^T + O_{12}O_{12}^T = 1 \quad O_{22}O_{22}^T + O_{21}O_{21}^T = 1 \quad O_{11}O_{21}^T + O_{12}O_{22}^T = 0 \tag{4.10}$$

and construct OU_0.

$$U = OU_0 = \begin{pmatrix} O_{11} + g_2 O_{12} b & g_2 O_{11} a + O_{12} \\ g_2 O_{22} b + O_{21} & O_{22} + g_2 O_{21} a \end{pmatrix}. \tag{4.11}$$

For U to have the same form as (4.2) we must choose O_{12} and O_{21} to be of order $O(g_2)$. This implies that up to corrections of order $O(g_2^2)$ we have

$$O_{11} = \mathbb{1} \qquad O_{22} = \mathbb{1} \qquad O_{21} = -O_{12}^T \tag{4.12}$$

and thus U becomes:

$$U = \begin{pmatrix} \mathbb{1} & g_2 a_0 + O_{12} \\ g_2 b_0 - O_{12}^T & \mathbb{1} \end{pmatrix}. \tag{4.13}$$

Thus, all matrices of type (4.2) are related by left-multiplication with orthogonal matrices, up to $O(g_2^2)$.

The most general solutions of equation (4.4) is:

$$\begin{aligned} &a_{iA} = \rho_{iA} c_{iA} + e_{iA} \qquad b_{Ai} = \sigma_{iA} c_{iA} - e_{iA} \\ &\rho_{iA} + \sigma_{iA} + 1 = 0\,. \end{aligned} \tag{4.14}$$

As stressed before, to pick a particular solution we require that $\hat{\gamma}_{iA}$ matches the string theory predictions for the matrix elements of the hamiltonian between a 1- and a 2-string state.

As before, due to the qualitative differences between $\mathcal{O}_{0,0}^{[\Phi]_{J_1}[\Psi]_{J-J_1}}$ and $\mathcal{O}_{n,\emptyset}^{[\Phi,\Psi]_{J_1}[\mathbb{1}]_{J-J_1}}$, it is necessary to treat $\hat{\gamma}_{ix}$, $\hat{\gamma}_{i\alpha}$ separately.

$$\begin{aligned}\hat{\gamma}_{ix} &= 0 - \delta_i(\rho_{ix}C_{ix} + e_{ix})\\ \hat{\gamma}_{i\alpha} &= -\frac{\lambda' g_2}{\pi^2\sqrt{J}}\sqrt{\frac{1-x}{x}}\frac{n_\alpha}{m_i x - n_\alpha}\sin^2\pi x m_i - (\delta_i - \delta_\alpha)(\rho_{i\alpha}C_{i\alpha} + e_{i\alpha})\,.\end{aligned} \tag{4.15}$$

We can now choose ρ and e such that $\hat{\gamma}_{ix}$ and $\hat{\gamma}_{i\alpha}$ match the string theory predictions [8]. As discussed above this is not a unique choice, but all such choices are related by orthogonal transformations.

$$\rho_{iA} = \sigma_{iA} = -\frac{1}{2} \qquad e = 0\,. \tag{4.16}$$

It is easy to see that all constraints are satisfied and the anomalous dimension matrix is:

$$\hat{\gamma} = \begin{pmatrix} 16\pi^2\lambda' m_i\delta_{ij} & -\frac{\lambda' g_2}{2\pi^2\sqrt{J}}\sin^2\pi x m_i & \frac{\lambda' g_2}{2\pi^2\sqrt{J}}\sqrt{\frac{1-x}{x}}\sin^2\pi x m \\ -\frac{\lambda' g_2}{2\pi^2\sqrt{J}}\sin^2\pi x m_j & 0 & 0 \\ \frac{\lambda' g_2}{2\pi^2\sqrt{J}}\sqrt{\frac{1-x}{x}}\sin^2\pi x m & 0 & 16\pi^2\lambda' n_\alpha\delta_{\beta\alpha} \end{pmatrix}. \tag{4.17}$$

Since the scalar product is diagonal, $\hat{\gamma}$ is the matrix representation of the generator of scale transformations and it matches the string theory prediction for the hamiltonian matrix. For later convenience let us list A:[6]

$$a_{iA} = b_{Ai} = -\frac{1}{2}c_{iA} \tag{4.18}$$

as well as the operator basis dual to string states at order g_2:

$$\hat{\mathcal{O}}_i = \mathcal{O}_i - \frac{1}{2}\sum_A C_{iA}\mathcal{O}_A \qquad \hat{\mathcal{O}}_A = \mathcal{O}_A - \frac{1}{2}\sum_i C_{iA}\mathcal{O}_i\,. \tag{4.19}$$

We have therefore succeded in reconstructing the string theory hamiltonian from pure gauge theoretic considerations. We identified the operator basis in which the matrix elements computed in gauge theory match the string theory computations in the oscillator number basis. This identification is not to be regarded as a prediction of gauge theory, since it was specifically constricted by comparing the matrix elements of the hamiltonian. However, now that the state-operator map is known to order g_2, all matrix elements between 1- and 2-string states of all operators should match between string and gauge theory. The discussion is, however, not finished yet.

4.2 Order g_2^2; the state - YM operator map; contact terms

In the previous section we found an operator redefinition which brought the anomalous dimension matrix to a symmetric form in the sector describing the mixing between operators dual to one and two-string states. However, analyzing the higher order corrections to

[6] Mixing of operators was discussed in [12] as well as in [13]. The precise formulae presented there are different from these ones. Our operator redefinitions seem to be related to those of [12] by an orthogonal transformation of the type described earlier in this section.

$\boldsymbol{\gamma}$ we are led to the conclusion that the anomalous dimension matrix should be symmetric to order g_2^2. Indeed, γ_{iA} receives corrections only at order g_2^3 while γ_{ij} receives corrections already at order g_2^2. The diagonalization discussed in the previous section also contributes terms of order g_2^2 to $\hat{\gamma}_{ij}$ and $\hat{\gamma}_{BA}$. These contributions are essentially fixed once the matrix U_0 is chosen. The corrections of order g_2^2 to γ_{ij} were computed in detail in [6] and [9]. Since the anomalous dimension matrix transforms as in (4.5) and γ_{ij} is already symmetric, it is a nontrivial check that $\hat{\gamma}_{ij}$ remains symmetric.

For the scalar product the situation is similar: the corrections to the 2-point functions of single- and double-trace operators are of order $O(g_2^3)$ while the corrections to the 2-point functions of single-trace operators are of order $O(g_2^2)$ and were computed in [6] and [9].

Since to this order C_{iA} is not modified at order g_2^2, it follows that the only modification that we are allowed to do to the previous analysis is to add terms of order g_2^2 to U_{ij} and U_{BA}. These additions are nevertheless uniquely fixed by the requirement that the scalar product be diagonal to order $O(g_2^2)$.

Let us now proceed with the computation. Using the results of [6] and [9] the scalar product is:

$$S = \begin{pmatrix} \delta_{ij} + g_2^2 \mu_{ij} & g_2 c_{iA} \\ g_2 c_{jB} & \delta_{BA} + g_2^2 M_{BA} \end{pmatrix} \tag{4.20}$$

with

$$\mu_{ij} = \begin{cases} \dfrac{1}{60} + \dfrac{21 - 2\pi^2 m_i^2}{48\pi^4 m_i^4} & m_i = m_j \neq 0 \\ \dfrac{105 + 8\pi^2 m_i^2}{384\pi^4 m_i^4} & m_i = -m_j \neq 0 \\ \dfrac{2\pi^2 (m_i - m_j)^2 - 3}{24\pi^4 (m_i - m_j)^4} + \dfrac{2m_i^2 - 3m_i m_j + 2m_j^2}{8\pi^4 m_i^2 m_j^2 (m_i - m_j)^2} & |m_i| \neq |m_j| \neq 0 \end{cases} \tag{4.21}$$

(we list only the cases when the single-trace operators are non-BPS since in the other cases the mixing with double-trace operators vanishes). Here we have made explicit the fact that the scalar product between single- and double-trace operators is of order g_2: $C_{iA} = g_2 c_{iA}$. The matrix M_{BA} has not been computed. We will determine it by requiring that $\hat{\boldsymbol{\gamma}}$ be symmetric in this sector as well. Then, modifying U_{ij} and U_{BA} with terms of order g_2^2, the matrix U and its inverse become:

$$U = \begin{pmatrix} 1 + g_2^2 d & g_2\, a \\ g_2\, b & 1 + g_2^2 \hat{d} \end{pmatrix}\,; \quad U^{-1} = \begin{pmatrix} 1 - g_2^2 (d - ab) & -g_2\, a \\ -g_2\, b & 1 - g_2^2 (\hat{d} - ba) \end{pmatrix}. \tag{4.22}$$

From the requirement that U diagonalizes S to order g_2^2 we find, besides equation (4.4), two new constraints:

$$\begin{aligned} \mu + d + d^T + ac^T + ca^T + aa^T &= 0 \\ M + \hat{d} + \hat{d}^T + bc + c^T b^T + bb^T &= 0\,. \end{aligned} \tag{4.23}$$

Let us analyze the consequences of these relations.

The first equation above uniquely determines the symmetric part of d (up to ambiguities described in the previous section). We find:

$$(d+d^T)_{ij} = -\mu_{ij} + \frac{3}{4}\sum_A c_{iA}c_{jA} = \frac{1}{2}\mu_{ij}\,. \tag{4.24}$$

The sum over A represents a summation over the discrete index labeling the doulble-trace opeartors n_A as well as an integration over $\frac{J_1}{J} = x \in [0,\,1]$.

Having determined the symmetric part of d we can now study $\hat{\gamma}_{ij}$ to order g_2^2.[7] A trivial computation reveals that

$$\begin{aligned}\hat{\gamma}_{ij} &= (\Delta_1)_{ji} + g_2^2[(\delta_j-\delta_i)d_{ij} + (\delta_i-\delta_A)a_{iA}b_{Aj} + a_{iA}\Delta'_{Aj} - \Delta_{iA}b_{Aj}]\\ \hat{\gamma}_{AB} &= (\Delta_2)_{AB} + g_2^2[(\delta_B-\delta_A)\hat{d}_{AB} + (\delta_A-\delta_i)b_{Ai}a_{iB} + b_{Ai}\Delta_{iB} - \Delta'_{Ai}a_{iB}]\,.\end{aligned} \tag{4.25}$$

In the first expression above a sum over A is implied as well as an integral over $x\in[0,1]$. These last expressions have a smooth limit $g_2\to 0$, since both Δ and Δ' are proportional to g_2. Also, $(\Delta_1)_{ij}$ and $(\Delta_2)_{ij}$ contain the order g_2^0 diagonal part $\delta_i\delta_{ij}$ and $\delta_A\delta_{AB}$ together with order g_2^2 entries. In the second line above a sum over i over all integers is implied.

Let us analyze in detail the terms of order g_2^2 in $\hat{\gamma}_{ij}$. We have to make sure that these terms lead to a symmetric anomalous dimension matrix, otherwise its interpretation as hamiltonian is not valid. This might present a problem, since (4.25) is certainly not manifestly symmetric.

We can proceed in various ways, including brute force computation, but it is more instructive to isolate the symmetric and antisymmetric parts of $\hat{\gamma}_{ij}$. We will use for a and b the expressions computed in the previous section, equation (4.18). It is trivial to see that the freedom of multiplying U_0 from the left by an orthogonal matrix survives at order g_2^2. Indeed, if $\hat{\boldsymbol{\gamma}} = U_0\boldsymbol{\gamma}U_0^{-1}$ were symmetric, then so is $\hat{\boldsymbol{\gamma}}_O = O\hat{\boldsymbol{\gamma}}O^T$.

We will write explicitly the antisymmetric part, leaving for the time being the symmetric part in terms of the matrix a. With these clarifications, contributions of order g_2^2 to $\hat{\gamma}_{ij}$ due to the diagonalization of the scalar product are:

$$\begin{aligned}\hat{\gamma}_{ij}\Big|_{g_2^2} &= -\frac{1}{2}(\delta_i-\delta_j)(d+d^T+ac^T+ca^T+aa^T)_{ij} + \frac{1}{2}(\Delta_{iA}c_{jA} - \Delta_{jA}c_{iA}) -\\ &-\frac{1}{2}(\delta_i+\delta_j-2\delta_A)a_{iA}a_{jA} + (\delta_j-\delta_i)(a_{iA}c_{jA} - a_{jA}c_{iA}) +\\ &+\Delta_{iA}\left(\frac{1}{2}c_{jA}+a_{jA}\right) + \Delta_{jA}\left(\frac{1}{2}c_{iA}+a_{iA}\right) + \frac{1}{2}(\delta_i-\delta_j)(d_{ij}-d_{ji})\end{aligned}$$

with the same provisions for the summation over A.

Let us analyze the possibly troublesome first line in the equation above. Using equation (4.23) as well as the explicit expression for Δ_{iA} from equation (3.53) we find that it becomes:

$$\frac{1}{2}(\delta_i-\delta_j)\mu_{ij} - 8\pi^2\lambda' g_2(m_i-m_j)\sum_n \frac{n_\alpha}{x}c_{i\alpha}c_{j\alpha} = 0\,. \tag{4.26}$$

[7] $\hat{\gamma}_{iA}$ is unchanged to order g_2^2 compared with the previous section.

Thus, the potential problematic antisymmetric contribution to the anomalous dimension matrix vanishes. Thus, we are therefore left with:

$$\begin{aligned}\hat{\gamma}_{ij}\Big|_{g_2^2} &= -\frac{1}{2}(\delta_i+\delta_j-2\delta_A)a_{iA}a_{jA}+(\delta_j-\delta_i)(a_{iA}c_{jA}-a_{jA}c_{iA})+\\ &\quad+\Delta_{iA}\left(\frac{1}{2}c_{jA}+a_{jA}\right)+\Delta_{jA}\left(\frac{1}{2}c_{iA}+a_{iA}\right)+\frac{1}{2}(\delta_i-\delta_j)(d_{ij}-d_{ji})\,. \end{aligned}\tag{4.27}$$

The sum over A and the integral over x can be easily computed and the result is:

$$\hat{\gamma}_{ij}\Big|_{g_2^2} = \frac{1}{2}(\delta_i-\delta_j)(d_{ij}-d_{ji})+B_{ij}\tag{4.28}$$

with B_{ij} given by:

$$B_{ij}=\begin{cases}0 & i=0 \;\text{ or }\; j=0\\ \frac{1}{3}+\frac{5}{2\pi^2m_i^2} & i=j\neq 0\\ -\frac{15}{8\pi^2m_i^2} & m_i=-m_j\neq 0\\ \frac{3}{2\pi^2m_im_j}+\frac{1}{2\pi^2(m_i-m_j)^2} & m_i\neq\pm m_j\neq 0\end{cases}\tag{4.29}$$

We interpret this as the contribution of the the contact terms on the string theory side to the matrix elements of the matrix elements of the hamiltonian between 1-string states.

The antisymmetric part of d is not fixed from the diagonalization of the scalar product. For $|m_i|\neq|m_j|$ we can define it such that the terms proportional to g_2^2 take any value, in particular it can be fixed such that the matrix elements of the hamiltonian between operators corresponding to 1-string states indeed matches the contribution of the contact terms, whatever that may be. This is however not the case if $m_i=\pm m_j$ since this implies that $(\delta_i-\delta_j)=0$. Thus, our analysis predicts that the contribution of the string theory contact terms to the matrix elements of the hamiltonian between 1-string states with equal masses is:

$$\begin{aligned}\langle m|H_{\text{contact}}|m\rangle &= \frac{1}{3}+\frac{5}{2\pi^2m_i^2}\\ \langle m|H_{\text{contact}}|-m\rangle &= -\frac{15}{8\pi^2m_i^2}\,.\end{aligned}\tag{4.30}$$

Now we return to $\hat{\gamma}_{AB}$ and the second equation (4.23). As in the case of $\hat{\gamma}_{ij}$ everything is in principle determined. While the matrix M describing the 2-point functions of double-trace operators is not known, consistency of our procedure determines it: we will require that the antisymmetric part of $\hat{\gamma}_{AB}$ cancels and from here we will find M_{AB}.

Using the results of the previous section we find for $\hat{\gamma}_{AB}$:

$$\begin{aligned}\hat{\gamma}_{AB}\Big|_{g_2^2} &= \frac{1}{2}(\delta_A-\delta_B)M_{AB}+\frac{1}{2}(\Delta_{iA}c_{iB}-\Delta_{iB}c_{iA})-\\ &\quad-\frac{1}{8}(\delta_A+\delta_B-2\delta_i)c_{iA}c_{iB}-\frac{1}{2}(\delta_A-\delta_B)(\hat{d}_{AB}-\hat{d}_{BA})\,,\end{aligned}\tag{4.31}$$

where we have separated the antisymmetric (first line) and symmetric parts (second line) and we have already used equation (4.23) to eliminate the symmetric part of $\hat{d}$.

Requiring that the antisymmetric part of $\hat{\gamma}$ vanishes determines all but the diagonal entries of M and the entries corresponding to at least one operator being a product of BPS operators. It is not hard to check that this equation is consistent for these cases as well. Consider for example $\delta_A = \delta_B$. In this case the first term manifestly vanishes while the second term vanishes only after the sum over i is performed. Making everything explicit we find the following expression for M:

$$M_{AB} = J\frac{g_2^2}{\pi^4}\sqrt{x_A^3 x_B^3(1-x_A)(1-x_B)}\sum_{m_i=-J/2}^{J/2}\frac{\sin^2\pi m_i x_A}{(m_i-\frac{n_A}{x_A})^2}\frac{\sin^2\pi m_i x_B}{(m_i-\frac{n_B}{x_B})^2}\left[\frac{m_i}{\frac{n_A}{x_A}+\frac{n_B}{x_B}}-1\right]. \tag{4.32}$$

As for single-trace operators, we can choose the antisymmetric part of $\hat{d}$ such that almost all off-diagonal entries of $\hat{\gamma}_{AB}$ vanish. Indeed, if $(\delta_A - \delta_B) \neq 0$ we can freely set to 0 the first term in (4.31) together with the corresponding entries of Δ_2. Leaving aside the case $A = -B$ (i.e. $n_\alpha = -n_\beta$) for which a separate computation must be done (and we will not do it here), we find that $\hat{\gamma}_{AA}$ is given by:

$$\hat{\gamma}_{AA} = (\Delta_2)_{AA} + \frac{g_2^2}{4}\sum_i(\delta_i - \delta_A)c_{iA}c_{iA}\,, \tag{4.33}$$

where we explicitly wrote the sum over i.

Now we analyze this case by case, i.e. we split the index A into x and α. Furthermore, the sum over i is rather difficult to perform. Since the summand is well-behaved at all points, we can replace the sum over i from $-J/2$ to $J/2$ by an integral over $z \in [-1/2,\, 1/2]$ and multiply by another factor of J. While this operation is valid up to corrections of order $1/J$, this approximation is enough to illustrate that the sum is subleading in $1/J$. We find:

$$\begin{aligned}\hat{\gamma}_{xx} &= (\Delta_2)_{xx} + \frac{g_2^2}{4}\sum_i \delta_i c_{ix}c_{ix} \\ &= (\Delta_2)_{xx} + \frac{4}{J\pi^2}\lambda\, g_2^2 \sum_{m_i=-J/2}^{J/2}\frac{m_i^2}{J^2}\frac{\sin^4\pi\frac{m_i}{J}J_1}{\frac{m_i^4}{J^4}}\frac{1}{J^4} \\ &= (\Delta_2)_{xx} + \frac{4}{J\pi^2}\lambda\, g_2^2\,\frac{1}{J^3}\int_{-1/2}^{1/2} dz\,\frac{\sin^4\pi z J_1}{z^2} \\ &= (\Delta_2)_{xx} + \frac{2}{J}\lambda'\, g_2^2\, x + O\left(\frac{1}{J}\right),\end{aligned} \tag{4.34}$$

where the extra factor of $\frac{1}{J}$ comes from the square of the coefficients c_{ix}.

The case $A = \alpha$ is treated similarly. We find:

$$\begin{aligned}\hat{\gamma}_{\alpha\alpha} &= (\Delta_2)_{\alpha\alpha} + \frac{4\lambda' g_2^2}{\pi^2 J^3}\frac{1-x}{x}\sum_{m_i=-J/2}^{J/2}\sin^4\pi\frac{m_i}{J}J_1\frac{\frac{m_i}{J}+\frac{n_\alpha}{J_1}}{\left[\frac{m_i}{J}-\frac{n_\alpha}{J_1}\right]^3} \\ &= (\Delta_2)_{\alpha\alpha} + \frac{2}{J}\lambda' g_2^2(1-x) + O\left(\frac{1}{J}\right).\end{aligned} \tag{4.35}$$

As before, the extra factor of $1/J$ comes from $c_{i\alpha}^2$.

Due to the extra $1/J$ factors, it follows that all corrections of order g_2^2 to the anomalous dimensions of double-trace operators vanish in the large J limit. On the string theory side this corresponds to a vanishing g_s^2 correction to the mass of 2-string states. This is indeed true, since these corrections come at 1-loop level from contact terms and thus are of order g_s^4.

5. Discussions

We have discussed in detail the realization of the order g_s corrections and some of the g_s^2 corrections to the string theory hamiltonian in the gauge theory dual. For the operators we were interested in, i.e. operators with two scalar impurities, we have successfully recovered the string theory prediction. Of crucial importance in recovering its matrix elements was the identification of an operator basis which is orthonormal to order g_s^2. While the mere fact that the string theory hamiltonian can be identified in gauge theory is not surprising, it is certainly a surprising result that its matrix elements are computable in perturbation theory. This extends to the interacting string theory the observation that its gauge theory dual develops small effective coupling.

The explicit computations presented in this paper hold to order $g_s = \lambda' g_2$ and match the string theory predictions to this order, in the limit $\mu \to \infty$. While on the string theory side it seems complicated to find the corrections of order $\frac{1}{\mu}$, this seems relatively straightforward in gauge theory, since they correspond to higher loop corretions at fixed genus. In particular, following the diagramatic expansion of the gauge theory, it is not hard to see that the γ_{iA} entries of the anomalous dimension matrix have the following generic form

$$\gamma_{iA} = \sum_{m\geq 0}\sum_{n\geq 0} a_{mn}(g_2\lambda')^{2n+1}\lambda'^m\,. \tag{5.1}$$

The identification of the string theory hamiltonian in the gauge theory allows one to approach this computation, as the string theory prediction is not know. The basis of operators found in this paper is the correct one for finding a_{0m} for all positive m.

The basis-independent way of stating our main result is that the Yang-Mills version of the string hamiltonian is the difference between the generator of scale transfomations and the $U(1)_R$ generator corresponding to the propagation of the plane wave. With hindsight, this result is not all that surprising. After all, this is a consequence of the AdS/CFT dictionary, with a slight twist introduced by the light-cone gauge: the generator of scale transformations is the gauge theory dual of AdS time translations while the $U(1)_R$ generator is the gauge theory dual of motions on one of the great circles on S^5. Whether this identification survives interactions depends on whether any of the global symmetry generators are broken at the quantum level. On the string theory side, $AdS_5 \times S^5$ was argued to be an exact solution of string theory. Thus, we do not expect its symmetry algebra to be modified. Therefore, the AdS/CFT correspondence implies that

$$P^- = \Delta - J \tag{5.2}$$

to all orders in the string coupling constant. Matching the matrix elements on both sides becomes therefore a matter of correctly identifying the string state - gauge theory operator

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map. However, this identification cannot be done without string theory input. One needs to pick some operator and fix the state operator map such that the gauge theory matrix elements agree with the string theory computation in the desider basis. This generically fixes the freedom left by the requirement that the scalar product of operators be diagonal. Then, the matching of the matrix elements of any other operator between gauge and string theory can be considered as a check of the duality at interaction level.

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A. Tree level 3-point functions

In this appendix we compute the tree-level 3-point functions of BNM operators.

- $\langle \bar{\mathcal{O}}_m^{[\Phi,\Psi]_J} \mathcal{O}_\Phi^{J_1} \mathcal{O}_\Psi^{J_2} \rangle$

  Start with

$$\mathcal{O}_\Phi^{J_1} \mathcal{O}_\Psi^{J_2} = \mathrm{Tr}\left[\Phi Z^{J_1}\right] \mathrm{Tr}\left[\Psi Z^{J_2}\right] \tag{A.1}$$

and project onto single trace operators. The second operator can be inserted between any two fields of the first operator and furthermore Ψ can sit at any position in the string of J_2 Z fields. Thus we find:

$$\frac{1}{N} \sum_{l=0}^{J_1} \sum_{n=0}^{J_2} \mathrm{Tr}[\Phi Z^l (Z^n \Psi Z^{J_2-n}) Z^{J_1-l}]\,. \tag{A.2}$$

Computing the scalar product between this and a single-trace operator $\mathcal{O}_m^{[\Phi,\Psi]_J}$ we find that it is given by the double sum:

$$\sum_{l=0}^{J_1} \sum_{n=0}^{J_2} e^{2\pi i (l+n)\varphi_m^J}\,. \tag{A.3}$$

Indeed, the l^{th} term in $\mathcal{O}_m^{[\Phi,\Psi]_J}$ is weighted with $\exp(2\pi l \varphi_m^J)$ and l is the number of Z fields between Φ and Ψ. This sum can be eqsily computed and it gives:

$$\sum_{l=0}^{J_1} \sum_{n=0}^{J_2} e^{-i(l+n)\varphi_m^J} = \frac{(1 - e^{-i(J_1+1)\varphi_m^J})(1 - e^{i(J_1-1)\varphi_m^J})}{(1 - e^{-i\varphi_m^J})^2}\,. \tag{A.4}$$

Thus, to leading order in a $J \to \infty$ expansion is:

$$\langle \bar{\mathcal{O}}_m^{[\Phi,\Psi]_J} \mathcal{O}_\Phi^{J_1} \mathcal{O}_\Psi^{J_2} \rangle = -\frac{g_2}{\sqrt{J}} \frac{\sin^2 \pi m x}{\pi^2 m^2} + \mathcal{O}\left(\frac{1}{J^2}\right) \qquad g_2 = \frac{J^2}{N}\,, \quad x = \frac{J_1}{J}\,. \tag{A.5}$$

- $\langle \bar{\mathcal{O}}_m^{[\Phi,\Psi]_J} \mathcal{O}_n^{[\Phi,\Psi]_{J_1}} \mathcal{O}_{vac}^{J_2} \rangle$
  Start with

$$\mathcal{O}_n^{[\Phi,\Psi]_{J_1}} \mathcal{O}_{vac}^{J_2} = \frac{1}{\sqrt{J_1 J_2}} \sum_{l=0}^{J_1} e^{2\pi i l \varphi_n^{J_1}} \operatorname{Tr}\left[\Phi Z^l \Psi Z^{J_1-l}\right] \operatorname{Tr}\left[Z^{J_2}\right] \tag{A.6}$$

and project onto single trace operators. The result is:

$$\begin{aligned} &\frac{1}{N\sqrt{J_1 J_2}} \sum_{l=0}^{J_1} e^{2\pi i l \varphi_n^{J_1}} J_2 \times \mathbb{P}_l \qquad \text{with} \quad \mathbb{P}_l \quad \text{given by} \\ &\mathbb{P}_l = (l+1) \times \operatorname{Tr}[\Phi Z^{J_2+l} \Psi Z^{J_1-l}] + (J_1 - l + 1) \operatorname{Tr}[\Phi Z^l \Psi Z^{J-l}] \,, \end{aligned} \tag{A.7}$$

where the overall factor of J_2 comes from the J_2 ways of choosing a Z in $\operatorname{Tr}[Z^{J_2}]$ and the factors of l and $(J_1 - l)$ in the bracket come from the number of ways of inserting Z^{J_2} in the first trace.

Computing the scalar product between this and the operator $\mathcal{O}_m^{[\Phi,\Psi]_J}$ we find:

$$\begin{aligned} &\frac{1}{N}\sqrt{\frac{J_2}{(J_1+1)(J+1)}} \sum_{l=0}^{J_1} e^{2\pi i l \varphi_n^{J_1}} \left[(l+1) e^{2\pi i (J_1 - l)\varphi_m^J} + (J_1 - l + 1) e^{-2\pi i l \varphi_m^J}\right] = \\ &= \frac{1}{N}\sqrt{\frac{J_2}{(J_1+1)\,(J+1)}} \left[\frac{\sin^2(\pi m x)}{\sin^2 \pi a} + J_1 + 2\frac{\sin(\pi m x)\cos\pi(mx + a)}{\sin \pi a}\right] \end{aligned} \tag{A.8}$$

where $a = \frac{m}{J} - \frac{n}{J_1}$ and $x = \frac{J_1}{J}$. The first two terms in the bracket above arise from the l and $J_1 - l$ terms while the last term comes from the 1 in each bracket. Keeping the leading term in the limit J, $J_1 \to \infty$, the final result is:

$$\langle \bar{\mathcal{O}}_m^{[\Phi,\Psi]_J} \mathcal{O}_n^{[\Phi,\Psi]_{J_1}} \mathcal{O}_{vac}^{J_2} \rangle = \frac{g_2}{\sqrt{J}} x^{3/2} \sqrt{1-x}\, \frac{\sin^2(\pi m x)}{\pi^2 (mx - n)^2} + \mathcal{O}\left(\frac{1}{J}\right) \qquad g_2 = \frac{J^2}{N}\,, \quad x = \frac{J_1}{J}\,. \tag{A.9}$$

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